The Chaos Hypertext Book

Mathematics in the Age of the Computer

Prefaces

The Chaos Hypertextbook

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0.1 What's New in The Chaos Hypertextbook

28 January 1999

- Moved to <hypertextbook.com> from <www.columbia.edu/~gae4>.
  Changed the name to "The Chaos Hypertextbook" from "Chaos, Fractals, Dimension".
- Made a cool new banner showing a Julia Set Cascade of the type explained in Chapter 2.3. Created this page and a page with links to this site.

7 June 1998

- Added a series of pages showing a zoom into the Mandelbrot set over 15 orders of magnitude.

19 February 1998

- Discovered the joys of server-side includes (SSI). Now pages have consistent colophons controlled by one file. If that doesn't mean anything to you, then forget about it.

15 December 1997

- Completely revised and updated all four chapters in html format. Gave this new website the name "Chaos, Fractals, Dimension".

10 January 1996

- Completed the text copy of the fourth chapter.

27 June 1995

- Completed the text copy of the third chapter.

12 June 1995

- Completed the text copy of the second chapter.

3 May 1995

- Completed the text copy of the first chapter.

Prefaces

The Chaos Hypertextbook

0.2 About The Chaos Hypertextbook

Who? You and I.

I wrote this book for anyone with an interest in chaos, fractals, non-linear dynamics, or mathematics in general. It's a moderately heavy piece of work, requiring a bit of mathematical knowledge, but it is definitely not aimed at mathematicians. My background is in physics and I use mathematics extensively in problem solving. Like many educated people, I also enjoy math as a diversion. This is the audience I am writing for.

What? Neat Stuff.

In the 1980s, strange new mathematical concepts burst forth from academic isolation to seize the attention of the public. Chaos. A fantastic notion. The study of the uncontainable, the unpredictable, the bizarre. Fractals. Curves and surfaces unlike anything ever seen in mathematics before. Surely, these topics are beyond the comprehension of all but the smartest, most educated, and most specialized geniuses. Wrong! Chaos, fractals, and the related topic of dimension are really not that difficult. One can devote an academic lifetime to them, of course, but the basic introduction presented in this book is no more difficult to understand than the straight line and the parabola.

When? Right Now.

Some of the topics discussed have roots extending back to the close of the Nineteenth Century. The really flashy stuff had to wait until integrated circuits integrated themselves into daily life. To attract the attention of the media-saturated you've got to have color, pattern, detail, and motion at a level beyond line drawings on paper. You need a computer. Actually, you need a lot of computers and they've got to be cheap, fast, and simple to operate so that many people will use them. You need to live at the dawn of the Twenty-first Century. If you're reading this text you have the tool needed to reproduce every image, movie, and data set found in this book. This is mathematics in the age of the computer.

Where? Nowhere and Everywhere.

This book can never exist on paper. Although copies of the linear text have existed on paper in the past and will again in the future, this is really a hypertext document. Move your finger over the linear text, press on a diagram or word and you leave behind a fingerprint. Move your cursor over the hypertext, press on a diagram or word and you're off viewing another page.

This book will never exist again as it does now. I intend to update and modify it on an irregular basis (that is, whenever I feel like it). Portions of this book were originally composed with Microsoft Word 5.1 running on a Macintosh LC. After Mosaic sparked the explosion of the World Wide Web in 1994, I knew that I would eventually transfer it to HTML. When the next tidal wave inundates the computer world, chances are this book will be washed away with it.

This book does not exist anywhere. There is no entity that contains it. I play with an edition of it that lives in my Power Macintosh, back it up on to Zip disks, download it to a server hidden somewhere on the planet, you access it, and copies of it bounce around the Internet until they land in your cache. This is not a book.

Why? Why Not.

I began writing *Chaos*, *Fractals*, *Dimension* because I was interested in the topics presented. After I saw how easy it all was to understand, it grew and grew until it covered over one hundred pages of double-spaced text. The augmented, hypertext version as it existed on 15 December 1997 was submitted as the final integrated essay for a Master of Science degree in secondary science education from Teachers College, Columbia University.

Expanded Table of Contents

Prefaces

1. **Mathematical Experiments**

   The first chapter introduces the basics of one-dimensional iterated maps. Say what? Take a function y = f(x). Substitute some number into it. Take the answer and run it through the function again. Keep doing this forever. This is called iteration. The numbers generated exhibit three types of behavior: steady-state, periodic, and chaotic. In the 1970's, a whole new branch of mathematics arose from the simple experiments described in this chapter.

2. **Strange & Complex**

   The second chapter extends the idea of an iterated map into two dimensions, three dimensions, and complex numbers. This leads to the creation of mathematical monsters called fractals. A fractal is a geometric pattern exhibiting an infinite level of repeating, self-similar detail that can't be described with classical geometry. They are quite interesting to look at and have captured a lot of attention. This chapter describes the methods for constructing some of them.

3. **What is Dimension?**

   The third chapter deals with some of the definitions and applications of the word dimension. A fractal is an object with a fractional dimension. Well, not exactly, but close enough for now. What does this mean? The answer lies in the many definitions of dimension.

1. General Dimension
2. Topological Dimension
3. Fractal Dimension

4. **Measuring Chaos**

The fourth chapter compares linear and non-linear dynamics. The harmonic oscillator is a continuous, first-order, differential equation used to model physical systems. The logistic equation is a discrete, second-order, difference equation used to model animal populations. So similar and yet so alike. The harmonic oscillator is quite well behaved. The paramenters of the system determine what it does. The logistic equation is unruly. It jumps from order to chaos without warning. A parameter that discriminates among these behaviors would enable us to measure chaos.

1. Harmonic Oscillator
2. Logistic Equation
3. Lyapunov Exponent
4. Lyapunov Space

Appendices

1. Printed Resources
2. Software Resources <-- unusually large page
3. Internet Resources <-- unusually large page

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0.3 Links to The Chaos Hypertextbook

- Art and Science, Dr Jean Debord
- Bob's Fractal Links, by Bob I assume
- Bookmarks, Erkki Kurenniemi, Heureka: The Finnish Science Center
- Bookmarks for Tomas B. Klos, University of Groningen
- Chaos Exercise Connections, J. Saunders
- Chaos Mathematics, Learning Fractal Geometry, Metaculture
- Chaos Theory, StudyWeb
- CSCI4446/6446 Course Materials for Spring 1999, Liz Bradley, University of Colorado
- Determinitve Chaos Software and related links, Stock Market Timing Using Advanced Mathematics
- Dynamical Sytems III (063091), Bob Johnson, University of Durham
- ED 265i -- Unit 4 Lecture -- An Anatomy of a Web Site, Kathy Rutkowski, George Washington University
- EDU2 : Level 2, E.J. Inglis-Arkell
- Enlasces a Otra Interesantes Paginas, José Luis De la Cruz Lázaro
- Enlaces y referencias sobre caos y fractales, Gonzalo Alvarez, Instituto de Física Aplicada
- Excite Education: Science & Nature: Mathematics: Fractals & Complexity
- Fractal & Fractal, Jiang Ching Kuen, Tsing Hua University, look under "¸H§Îªº*Æ*Ç"

- Fractal Links, MathsNet, Anglia Multimedia
- Fractal Links on Paul N. Lee's Website, look for "various software"
- Fractal Top, Garr Lystad
- FractalTrees, Simon Woodside, University of Waterloo
- Fractal Web Sites, Fractal Domains, Dennis C. De Mars
- Fractals - Links, J.P. Louvet, Université Bordeaux
- Frédéric Paccaut, Universite de Bourgogne
- Fun Science Stuff, Bellarmine College
- General Links, Kevin Raulerson
- I Jornada de Física y Matemática
- Ideas and Activities, Vicki F. & Richard M. Sharp, California State University, Northridge
- Interessante links, Educatieve Faculteit Amsterdam
- Investigating The Mathematics of Complexity
- Juan P. Cerezo's Bookmarks, La Universidad Autónoma de Madrid
- Kaosplock, Lars Rosenberg
- Knot A Braid of Links, Canadian Mathematical Community (CAMEL)
- Kuo-Chang Chen's Homepage, University of Minnesota

- L'art et la science, Dr Jean Debord
- lec1, Guy Zimmerman, Techno-Z Fachhochschule Salzburg
- Lecture for Feb 03, 1999 - Physics 351, Larry Gladney, University of Pennsylvania
- Links, Institut für Physiologie, Freie Universität Berlin
- Links, Math Teacher's Home Page, Toronto District School Board
- Links on Chaos and Links on Chaos, Chaos 21, Inje University
- LookSmart / World - Library - Sciences - Math - Fractals & Complexity
- Fractals & Complexity, Looksmart
- Math (question), Hong Kong School Net

- M@ths: Fractals! Diana Rolf
- Mandelbrot Exhibition, Jonathan Bowen, Oxford University
- Mandelbrot Set, Dale Winter, University of Michigan
- Math 335 Chaos, Fractals and Dynamics, Randall Pyke, University of Toronto
- Math 357 Complex Functions, Dr. John Maharry, Franklin College
- Math, Jason Vestuto, University of Maryland
- Math Forum Internet Resource Collection, linked on numerous pages in the index
- Math Sites, South Fayette Township School District
- Mathematics, Mason Library Information System, Keene State College
- Mathematics, Todd R. Shaw, University of Utah
- Mathematics, Science and Engineering, Ralph Carmichael
- Mathpuzzle.com, Ed Pegg Jr,
- Molecular Modeling, Michael C. Tims, University of Maryland
- Netscape Search: Science: Math: Chaos
- NetWatch (19 February 1999), American Association for the Advancement of Science (AAAS)
- Nonlinear Dynamics, Mathematics Archives, University of Tennesee, Knoxville
- Nonlinear Sites, Chaos at Maryland, University of Maryland
- Nonlinearity and Complexity Home Page, Moses A. Boudourides, Democritus University of Thrace
- Oton kirjanmerkit, Otto Hyvärinen
- Physics 4267 - Introductory Nonlinear Dynamics & Chaos, Georgia Institute of Technology
- Physics and astronomy links, Aarhus Universitet
- PouĨitá literatura, Vít Prudil, MasarykUniversity
- Principia - Internet, Eduardo René Rodríguez Ávila

- Recursos en la Web, La teoría del Caos, Author Unknown
- References on Chaos and Fractals, PMATH 399c, Will Gilbert, University of Waterloo
- Referencias, A. Carden, Universidad de Los Andes
- School of Physics Links, Georgia Institute of Technology
- sci.nonlinear FAQ, James D. Meiss (also available at mirror sites)
- Science, Physics, Relativity, Faq's and Feedback
- Scott's Nonlinear Science Hotlist, Scott Peckham, University of Colorado
- Spanky: What's New, Noel Giffin, TRIUMF
- Spring High School Math Links, Spring, Texas
- Sprott's Spots Awards and Most Popular Referrers, J. C. Sprott, University of Wisconsin, Madison
- Stock Market Timing, Galateia Corp.
- Strange Attractors, John Starrett, University of Colorado, Denver
- Telson Spur: Field Nodes: Chaos & Complexity; Math, Phil Hughes
- Topology, BUBL LINK, Strathclyde University
- Tutorial Links, CALResCo (Complexity and Artificial Life/Alife Research)
- W4T Undergraduate Analysis, Using the Web to Teach Mathematics, University of Colorado at Denver
- Homework Helpers - Math and Homework Helper and Reference Internet Sites, Michael Schelling
- Very Best of Le WEB non linéaire, Bibliothèque Universitaire du Havre
- Web Bytes, National Council of Teachers of Mathematics
- Web resources for MATH 1580/1590, Martin Muldoon, York University
- Web Directory, Grand Junction Sentinel & Longview News-Journal
- What are fractals? Giuseppe Zito, Università e Politecnico di Bari
- World of Mathematics, Han de Bruijn, Delft University of Technology

- Yahoo! Science: Mathematics: Chaos (also available at mirror sites)

Links to the Old Site

- Armin Leichtfuß, Analysis of Complex Systems, Technische Universität Darmstadt
- Chaos & Teria chaosu a Alife, Julius Csonto, Technickej univerzite v Kosiciach
  no reliable contact info for author
- Chaos Bookmarks for Joàko Poljak, Fakultet elektrotehnike i raãunarstva
- Complex Systems: Tutorials - Lectures, Univesity of Macedonia
- Graphing Non-Linear Equations, Kelleen Farrell
- LMC #1's Bookmarks, Nancy Moore, Base Line Middle School, Boulder Valley, Colorado
- Math Education Links, Faculty of Education, University of Western Ontario
- Mathematics Methods 6-12, University of Illinois at Springfield, Larry D. Stonecipher
- sci.fractals Meta-Index, Michael C. Taylor and Jean-Pierre Louvet (also available at mirror sites)
  last updated 8 March 1998
- Tomasz Sikora's Bookmarks, Nicolaus Copernicus Univeristy, Torun
- Views of the Mandelbrot Set from Galifrey, Keith & Fran
- Zone Newsletter - Issue 44, The Homeschool Zone

The European Mirror Site that Never Was

- http://link.medinn.med.uni-muenchen.de/cybermed/nonlin/
  203 Provisional Information

Mathematical Experiments

The Chaos Hypertextbook

1.1 Iterations & Orbits

One of the great breakthroughs in mathematics in this century has been the realization that even the simplest of dynamical systems may behave extremely unpredictably. Take the real functions

$$y = x^2 + c \quad \text{and} \quad x = y.$$

Many high school students and even some in junior high could successfully identify them as equations for a parabola and straight line respectively. There are few equations simpler than these and yet from them we can generate some rather complex and interesting behavior. Much of today's popular mathematics can be illustrated with these two equations (the kind mentioned in movies or television, often in passing and often in mutated form).

One way to interpret the functions is as two curves in the plane. A second way is as a series of instructions.

1. Given some number "x", take its square and add the constant "c". Call the result "y".
2. Given "y", do nothing. Call the result "x".
3. Repeat step 1 with the value found in step 2.

The first two instructions together form a mapping of one number on to another:

$$f: x \dashrightarrow x^2 + c.$$

If we did this for all real numbers we would be mapping the real numbers on to themselves:

$$f: R \dashrightarrow R.$$

The addition of the third step results in an iterated mapping. We will use the symbol $f^n(x)$ to represent the nth iterate of our original value "x". The instructions tell us to generate a series of numbers

$$x, f(x), f^2(x), f^3(x), \ldots, f^n(x), \ldots$$

which we will call the orbit. The initial value "x" is called the seed of the orbit. Note that there is no instruction telling us when to stop. Thankfully, humans are not quite so stupid as are the instructions given to them. Somewhere in this sequence, a pattern will emerge that will allow us to stop iterating and make a judgment. If it takes us too long to arrive at a decision or if we just get tired of doing all the work we can always turn it over to a computer. Let us now look at the behavior of some orbits.

The parameter c = 0 is by far the easiest to deal with as

$$f: x \dashrightarrow x^2 + c$$

will yield the following results:

$$f^n \to \begin{cases} \infty & |x| > 1 \\ 1 & |x| = 1 \\ 0 & |x| < 1 \end{cases} \qquad \text{as} \qquad n \to \infty$$

All orbits approach either zero or infinity except for those with seed x = +/-1. The points zero and infinity are called attracting fixed points or sinks because they attract the orbits of the points around them while +/-1 are called repelling fixed points or sources for the opposite reason.

When c > 1/4 the parabola is entirely above the diagonal line and all seed values will be driven off to infinity. When c = 1/4 the parabola and the diagonal line intersect at 1/2. Seeds with absolute value greater than 1/2 will expand off to infinity while those in the interval 1/2 <= |x| <= 0 approach 1/2 asymptotically. After about 700 iterations these seeds will have reached 4.99. The results of the first ten iterations for some seed values are presented in the table below.

The orbits of six seeds under the mapping f: x --> x^2 + 1/4
Note: the orbits not driven off to infinity are attracted to +1/2

+/-1	**+/-0.75**	**+/-0.5**	**+/-0.25**	**+/-0.1**	**0**
+1.25	+0.812	+0.5	+0.3125	+0.26	+0.25
+1.812	+0.910	+0.5	+0.3476562	+0.3176	+0.3125
+3.535	+1.078	+0.5	+0.3708648	+0.3508697	+0.3476562
+12.747	+1.412	+0.5	+0.3875407	+0.3731096	+0.9708648
+162.744	+2.246	+0.5	+0.4001878	+0.3892107	+0.3875407
+26485.994	+5.296	+0.5	+0.4101503	+0.4014850	+0.4001878
+701507907	+28.297	+0.5	+0.4182232	+0.4111902	+0.4101503
+4.921e+17	+800.985	+0.5	+0.4249107	+0.4190774	+0.4182232
+2.421e+35	+64158.262	+0.5	+0.4305491	+0.4256258	+0.4291070
+5.864e+70	+4.116e+11	+0.5	+0.4353725	+0.4311573	+0.4305491
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
infinity	infinity	+0.5	+0.5	+0.5	+0.5

The fixed point changes as the parameter changes, falling ever so slightly as "c" becomes smaller. The parabola and the diagonal line intersect at two points now; the roots of the equation

$$x^2 + c = x.$$

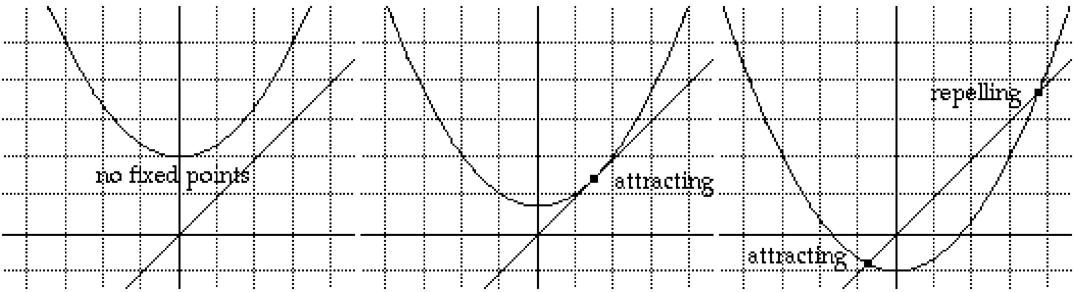
Upon further analysis it can be shown that the smaller of the two roots is an attracting fixed point and the larger of the two roots is a repelling fixed point. We have already shown this for the special case of c = 0 so let's try with another easy value. When the parameter c = -3/4 the equation now has roots of -1/2 and +3/2. The results of the first ten iterations for some seed values are presented in the table below.

The orbits of six seeds under the mapping f: x --> x^2 - 3/4
Note: the orbits not driven off to infinity are repelled from +3/2 and attracted to -1/2

+/-1.75	**+/-1.5**	**+/-1**	**+/-0.75**	**+/-0.5**	**+/-0.25**
+2.31	+1.5	+0.25	-0.1875	-0.5	-0.6875
+4.59	+1.5	-0.6875	-0.71484375	-0.5	-0.2773437
+20.38	+1.5	-0.2773437	-0.2389984	-0.5	-0.6730804
+414.93	+1.5	-0.6730804	-0.6928797	-0.5	-0.2969627
+172173.29	+1.5	-0.2969627	-0.2699176	-0.5	-0.6618131
+2.964e+10	+1.5	-0.6618131	-0.6771444	-0.5	-0.3120033
+8.787e+20	+1.5	-0.3120033	-0.2947537	-0.5	-0.6525639
+7.721e+41	+1.5	-0.6525639	-0.6650421	-0.5	-0.3240428
+5.962e+83	+1.5	-0.3240428	-0.3077189	-0.5	-0.6449962
overflow	+1.5	-0.6449962	-0.6553090	-0.5	-0.3339798
. . .	. . .	. . .	. . .	. . .	. . .
infinity	+1.5	-0.5	-0.5	-0.5	-0.5

As we suspected, +3/2, the larger of the two roots is indeed a repelling fixed point, but is -1/2 an attractor? The answer is yes, if you wait long enough. The orbits approach -1/2 by oscillating between two values on either side of -1/2, each of which approaches this value asymptotically. This behavior can be seen in the table above and is an indication of things to come.

Fixed points for some parameter values



c > 1/4 c = 1/4 -3/4 < c < 1/4

For values of the parameter c < -3/4, orbits that formerly would have approached the smaller of the two roots now oscillate between two distinct values. The attracting fixed point has bifurcated or split and the orbit is no longer stable but periodic, alternating between two values. As "c" becomes ever more negative another bifurcation takes place and the period doubles to 4 and then again to 8, then 16, then 32, 64, *ad infinitum*. The distance between successive bifurcations, however, approaches zero and does so in such a way that the period-doubling reaches infinity at a finite parameter value of about c < -1.4. Beyond this value orbits that were formerly periodic now wander over an aperiodic orbit about some finite interval within [-2, 2] and will visit every region of this interval. Such behavior is said to be ergodic and is a characteristic of chaos. In addition, seed values which are initially close to each other will, after a few iterations, follow orbits that are wildly different. This behavior, which exhibits sensitive dependence on initial conditions, is said to be chaotic and the values of the parameter "c" over which such behavior occurs is called the chaotic regime. The sequence of bifurcations leading up to the chaotic regime is known as the period-doubling route to chaos.

If this is getting to wordy, perhaps you should read on. In the sections following, these descriptions are rendered graphically.

Mathematical Experiments

The Chaos Hypertextbook

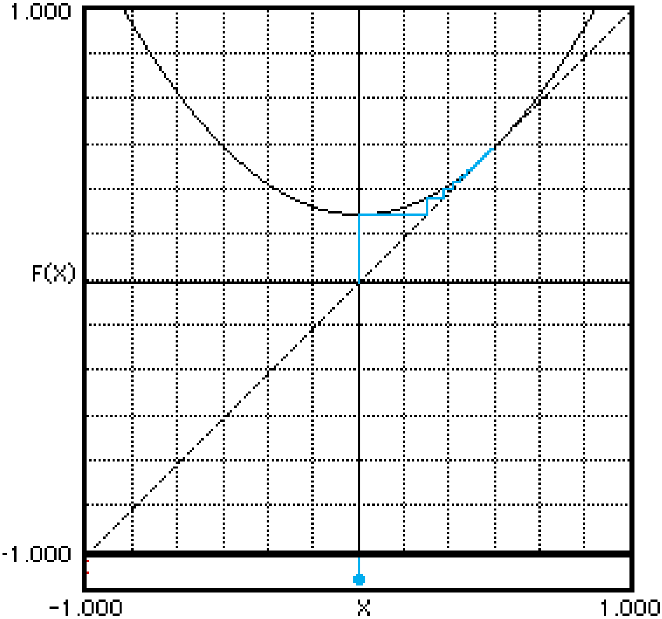
All diagrams drawn with 1-D Chaos Explorer

1.2 Orbits & Bifurcations

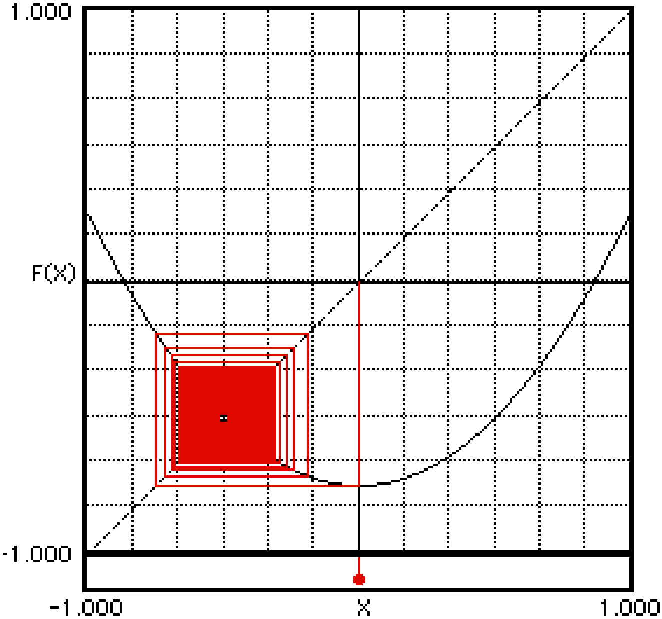
A more intuitive approach to orbits can be done through graphical representation using the following rules:

1. Draw both curves on the same axes. Pick a point on the x-axis. This point is our seed.
2. Draw a vertical straight line from the point until you intercept the parabola.
3. Draw a horizontal straight line from the intercept until you reach the diagonal line.
4. Repeat step 2 with this new point.

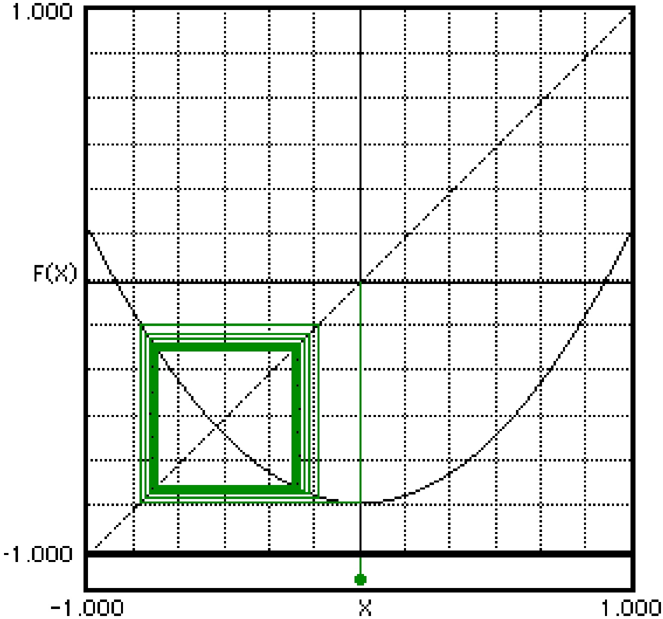
The following is a series of graphs detailing some of the behaviors described earlier. Because of their appearance, these diagrams are commonly known as web diagrams (or cobweb diagrams).



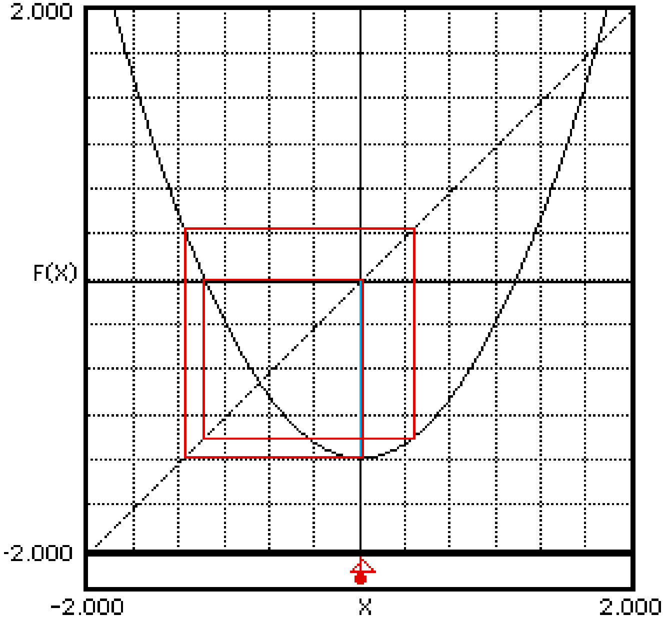
This graph shows the simple fixed-point attractive behavior of the parameter value c = 1/4 for the seed value of 0. Zero will be used as a standard seed for all further diagrams because it is "well-behaved". Note how the orbit moves towards 1/2. Further examination shows this approach to be asymptotic.



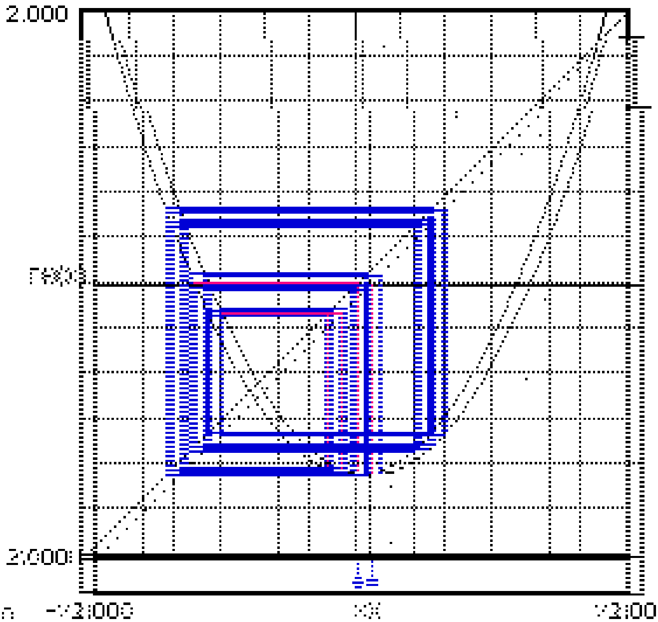
In this graph, the parameter value was set at c = -3/4. Note how the orbit approaches the fixed-point attractor from opposite sides. After more than 1000 iterations there is still a visible hole in the center. The orbit hasn't yet reached its final value.



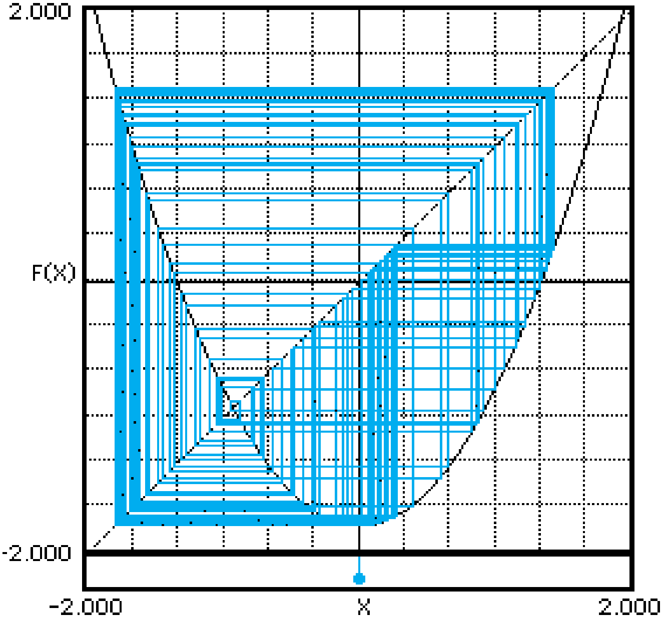
When c = -13/16 the orbit settles into a two-cycle, alternating between -3/4 and -1/4.



Here we see a four-cycle. When c = -1.3, the orbit oscillates over the values -1.2996224637, 0.3890185483, -1.1486645691, and 0.0194302923. This one settles down rather quickly. After only 100 iterations, it already looked complete.



This orbit was drawn using a parameter value of c = -1.4015. Although it looks similar to the previous diagram, the iterates never seem to repeat. Instead, they slosh around within bands. Tiny adjustments in initial conditions give orbits that are obviously different. At c = -1.4, the orbit had a period of 32, now the orbit has a period of infinity.

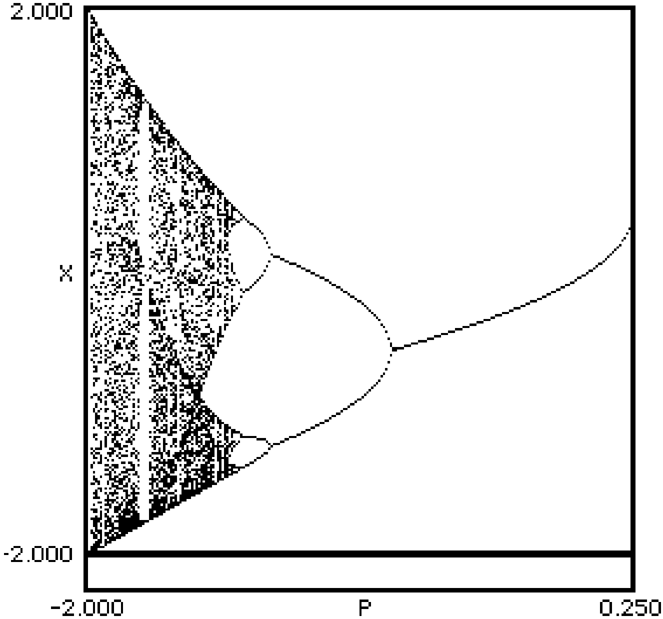


If this isn't chaos, I don't know what is. At c = -1.8, the orbit covers every region of some subinterval of [-2, 2]. This picture shows just a small subset of all the points the orbit will eventually visit.

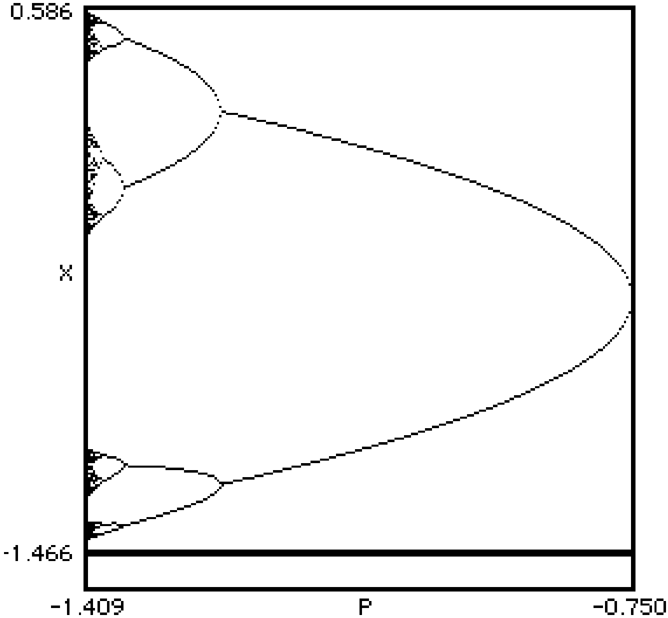
A way to see the general behavior of the mapping

$$f: x \dashrightarrow x^2 + c$$

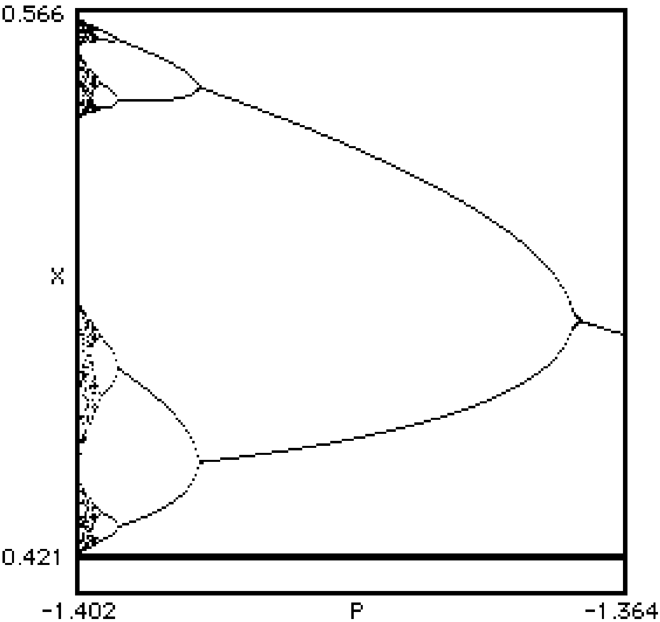
is to plot the orbits as a function of the parameter "c". We will not plot all the points of an orbit, just the most indicative ones. The first several hundred iterations will be discarded allowing the orbit to settle down into its characteristic behavior. Such a diagram is called a bifurcation diagram as it shows the bifurcations of the orbits (among other things).



Here we see the full bifurcation diagram. Parameter values outside of the range [-2, 1/4] were not included as all of their orbits go to off infinity. Note how the single attracting fixed point bifurcates repeatedly and then becomes chaotic. Note also the window at c = -1.8. Let's examine these areas in more detail.



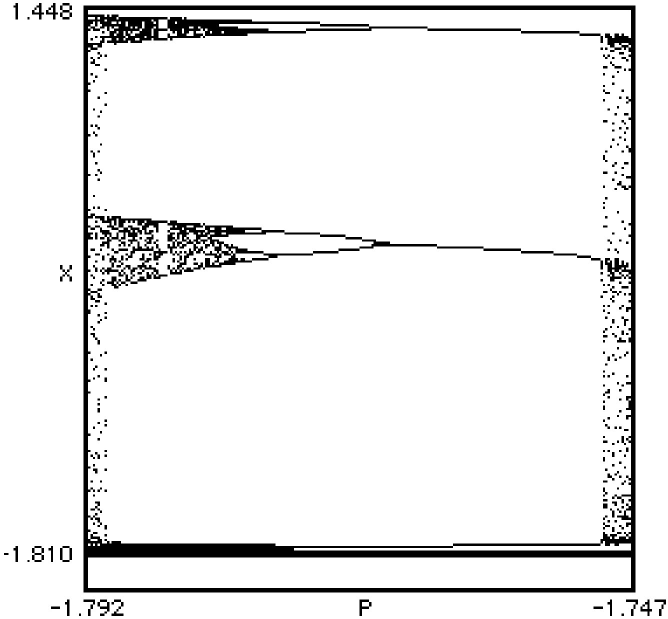
Here we see a magnification of the period-doubling region. Note successive bifurcations.



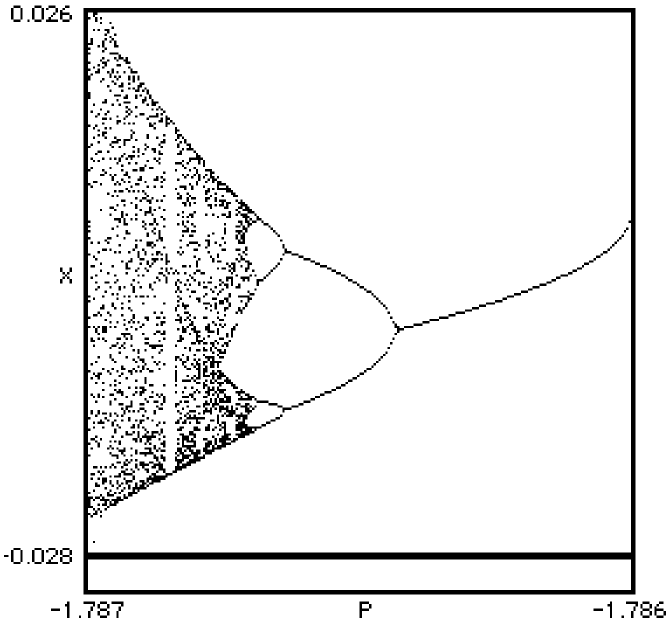
Zooming in on the region in the upper left-hand corner we see a repeat of the large scale structure. The period-doubling region exhibits self-similarity, that is, small regions look similar to large regions. This property can be seen in other parts of the diagram.

bifurcation

Here we see a magnification of the chaotic regime. Note the windows of periodicity amidst the chaos. Let's zoom in on the largest.



The structure of the window repeats the structure of the overall bifurcation diagram. The period doubling regime is the same but multiplied by three; that is, 3, 6, 12, 24, 48... instead of 1, 2, 4, 8, 16.... Note the window inside each lobe. The perspective is a bit whack as the window covers a region that is taller than it is wide.



This is a magnification centered on the center lobe of the largest window in the center lobe. Note the scale. We have zoomed in 1000 times. This diagram looks astonishingly similar to the original. The more things change the more they stay the same.

Mathematical Experiments

The Chaos Hypertextbook

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Bifurcation diagrams drawn with 1-D Chaos Explorer

1.3 Universality

As was shown in the diagrams, subregions within the bifurcation diagram look remarkably similar to each other and to the diagram as a whole. This self-similarity was shown to repeat itself at ever finer resolutions. Such behavior is characteristic of geometric entities called fractals (a topic I will address in the second chapter) and is quite common in iterated mappings. In the period-doubling region, for instance, the whole region beginning at the first bifurcation (called lambda$_1$) looks the same as either region beginning at the second bifurcation (lambda$_2$) which looks the same as either region beginning at the third bifurcation (lambda$_3$), and so on. Interestingly enough, the distance between successive bifurcation points (lambda$_n$) shrinks geometrically in such a fashion that the ratio of the intervals

$$\text{delta} = \frac{\Lambda_n - \Lambda_{n-1}}{\Lambda_{n+1} - \Lambda_n}$$

approaches a constant value as "n" approaches infinity. This constant, called Feigenbaum's number, crops up repeatedly in self-similar figures and has an approximate value of

> 4.
> 669201609102990671853203820466201617258185577475768632745651
> 343004134330211314737138689744023948013817165984855189815134
> 408627142027932522312442988890890859944935463236713411532481
> 714219947455644365823793202009561058330575458617652222070385
> 410646749494284981453391726200568755665952339875603825637225
> (Pickover 249)

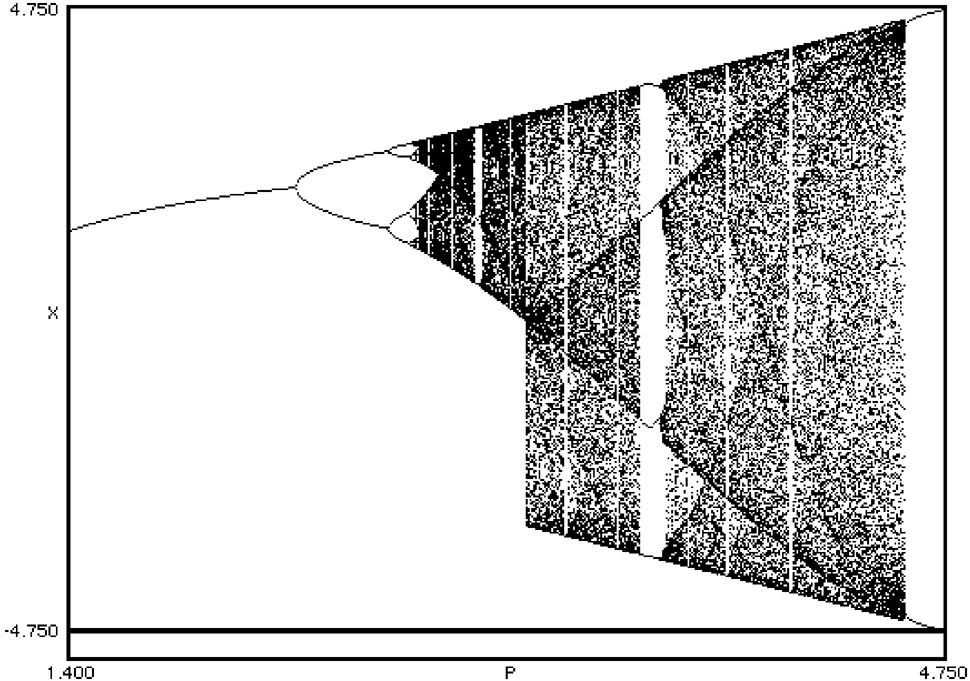
Not only does Feigenbaum's constant reappear in other figures, but so do many other characteristics of the bifurcation diagram. In fact, remarkably similar diagrams can be generated from any smooth, one-dimensional, non-monotonic function when mapped on to itself. A circle, ellipse, sine, or any other function with a local maximum will produce a bifurcation diagram with period-doublings who's ratios approach "delta". Together with a second constant "alpha", the scaling factor "delta" demonstrates a universality previously unknown in mathematics: metrical universality. The behavior of the quadratic map is typical for many dynamical systems. The the period-doubling route to chaos and the constants "alpha" and "delta" appeared in an unruly mess of equations used to describe hydrodynamic flow (Hofstadter). The realization that a set of five coupled differential equations describing turbulence could exhibit the same fundamental behavior as the one-dimensional map of the parabola on to itself was one of the great breakthroughs in late twentieth century mathematics.

This chapter has been devoted to the exploration of the simple one-dimensional iterative mapping

$$f: x \dashrightarrow x^2 + c$$

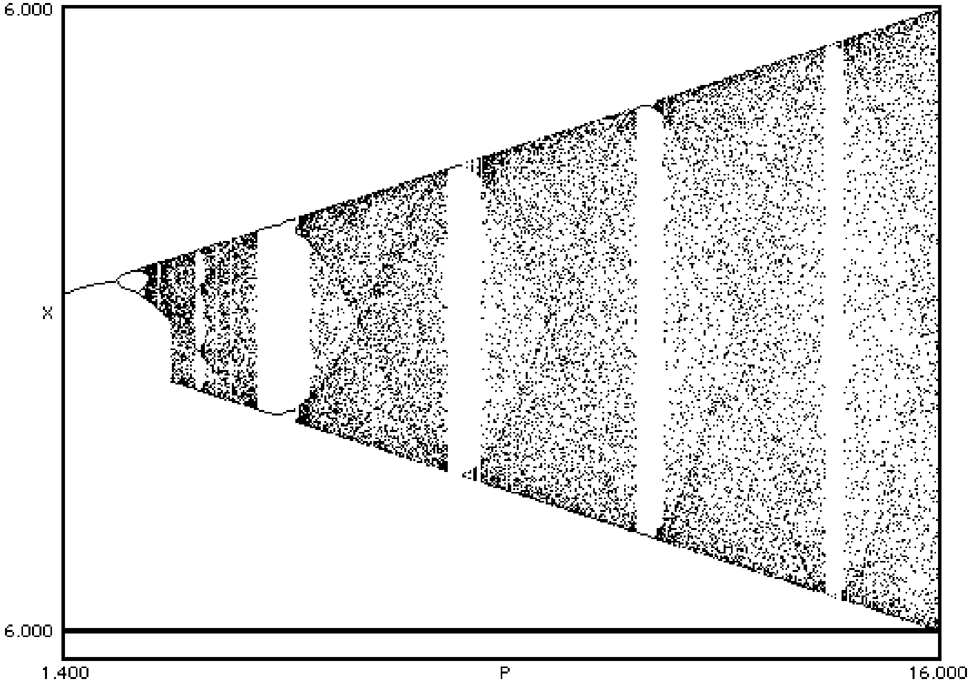
where "x" and "c" were real numbers. The statement was made that the behavior of this system is typical for "any smooth, one-dimensional, non-monotonic function when mapped on to itself." Many books on chaos mention how Feigenbaum's constants and the period-doubling route to chaos appear in other one-dimensional mappings, but few provide examples. Thus, I felt it necessary to explore the behavior of additional functions to see if such results were really universal. The results were rather interesting and quite unexpected. Let's look at the bifurcation diagrams for some other mappings.

Bifurcation diagram for
f: x --> c sin x



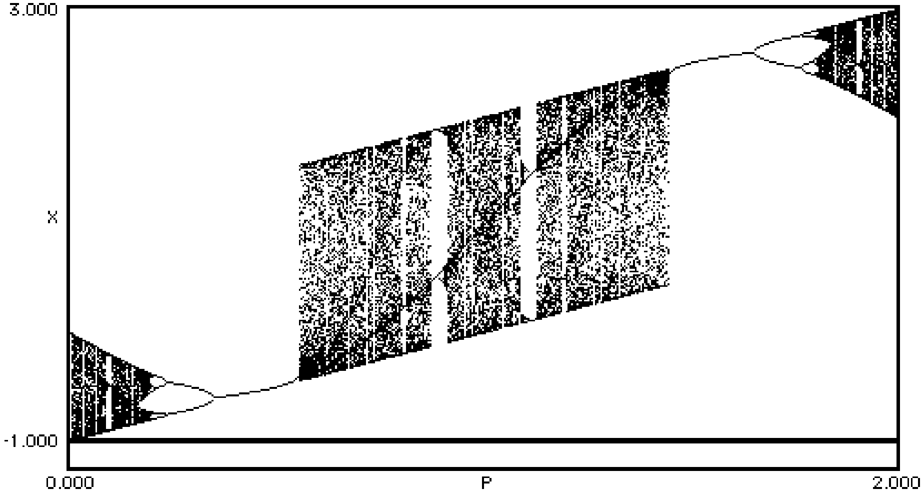
The structure of the bifurcation diagram is similar to that of f: x --> x^2 + c with a period-doubling path to chaos and an ergodic region with odd-period windows. In the quadratic case, the bifurcation diagram was finite and ended at a parameter value beyond which all orbits escaped to infinity. In the sinusoidal case, however, the map continues with an abrupt bump in the chaotic regime. This region is punctuated with *even*-period windows; the most prominent being a four-cycle. The chaotic regime widens and then terminates on a two-cycle, each side of which bifurcates. Again, the ergodic region bumps out and the pattern repeats itself *ad infinitum* as shown below.

Bifurcation diagram for
f: x --> c sin x
on a larger interval



As a variation on a theme, I tried the mapping f: x --> sin x + c. This is more like the quadratic mapping in that the parameter raises and lowers the function without changing its shape. As expected, the orbits gave a bifurcation diagram nearly identical to that for the quadratic map, but with a bit of a twist.

Bifurcation diagram for
f: x --> sin (pi x) + c

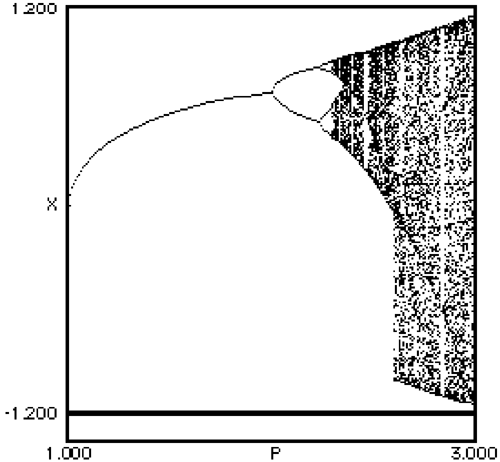


Again, we see an abrupt change from behavior characteristic of the quadratic map to a broad ergodic region. As with the previous sinusoidal map, this extra ergodic region has windows of even periodicity. (Note how the largest windows are open on one side.) This region ends with another quadratic-like bifurcation diagram rotated 180° from the first. The behavior of the diagram on the interval [2, 4] is identical to that on [0, 2] only shifted two units higher. Thus, the interval [0, 2] is characteristic of the remainder of the parameter values and can be used as a unit cell. The full diagram runs diagonally through the origin across the parameter space from negative infinity to positive infinity

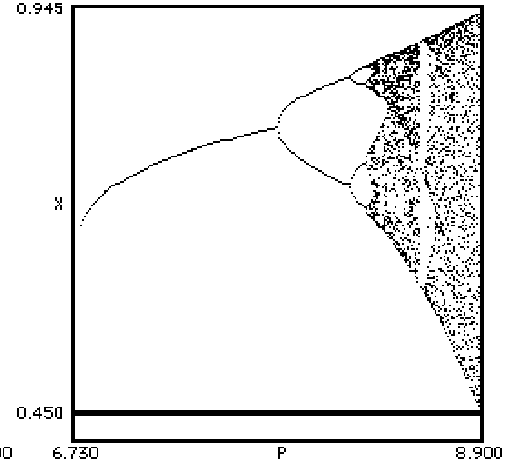
Some more mundane bifurcation diagrams are shown below. All functions are smooth and non-monotonic. Note that each one undergoes a period-doubling route to chaos and that there are always windows of odd periodicity within the chaotic regime.

Some more bifurcation diagrams

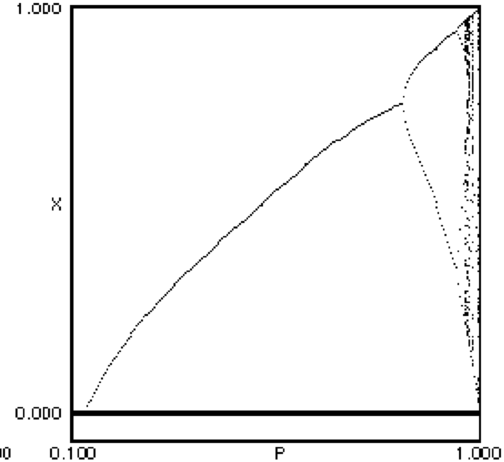
1.3 Universality



x --> cx (1 - x^2)

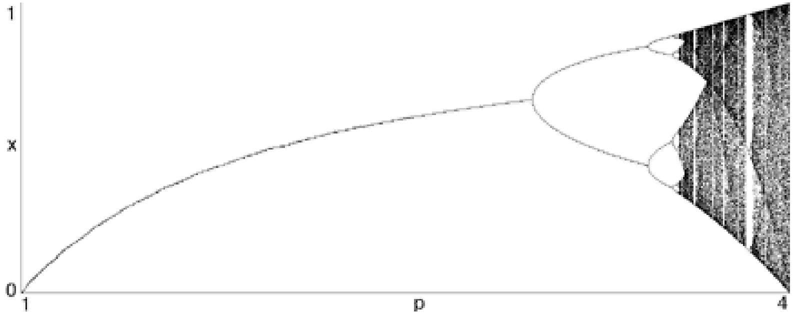


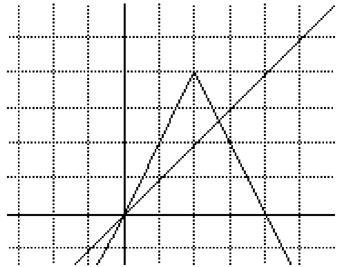
x --> cx^3 (1 - x)



x --> c (1 - $(2x - 1)^4$)

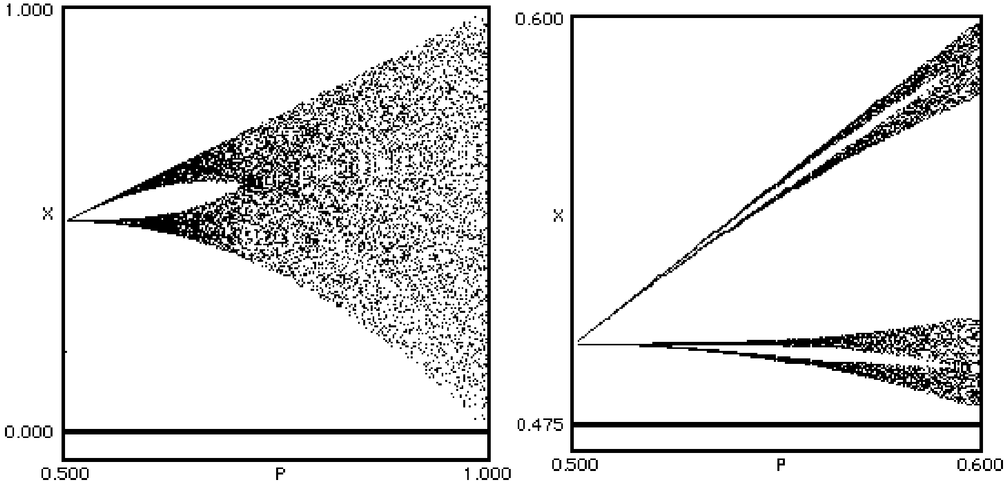
Bifurcation diagram of the logistic equation
x --> rx (1 - x)
This function is discussed extensively in the forth chapter
Click the image to view a larger copy

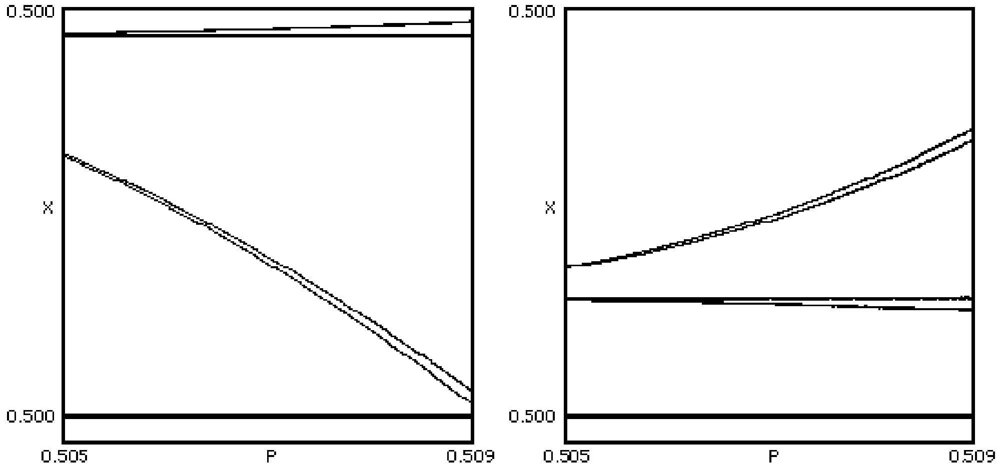




How about a non-monotonic function that is not smooth? The function f: x --> c (1 - 2 abs(x - 1/2)) is called the tent function for obvious reasons. For parameter values in the interval [0, 1/2] all well-behaved orbits collapse to zero while parameters greater than one drive all orbits to infinity. The first diagram below shows the orbits over the range [1/2, 1] where the behavior can be called "interesting". While there is a superficial similarity to many of the previous mappings (the two ergodic "arms" merging into one for example) this one does not exhibit period-doubling or windows. What look like single lines turn out to be, on closer examination, pairs of lines. When these are examined in more detail, they also turn out to be pairs of lines and so on. Orbits in the non-ergodic regions are not periodic but tend to cluster together and thus appear to have even periodicity. There is no bifurcation with this map. Those orbits that are neither stable nor ergodic most likely form a Cantor set. I have yet to find any mention of the surprisingly odd behavior of this map in the popular literature.

Bifurcation diagram for the tent function
f: x --> c (1 - 2 abs(x - 1/2))
showing successive magnifications




0.500
X
0.500
0.505
P
0.509
0.500
X
0.500
0.505
P
0.509

Strange & Complex

The Chaos Hypertextbook

Strange attractors drawn with Orbit and Mandella

2.1 Strange Attractors

The next logical step is to extend the concepts of iterated systems to multiple-dimensional mappings. Given the results of the previous explorations one can assume that much of the behavior found in the quadratic map will have its analog in higher dimensions and thus we need not introduce an entirely new vocabulary.

Let

$$f: R^n \dashrightarrow R^n$$

be a mapping of ordered n-tuples of real numbers on to themselves. Take the results and feed them back into the map repeatedly. For convenience sake, let's call the n-tuples points and R^n a space of dimension "n". This procedure generates an orbit of points

$$p, f(p), f^2(p), f^3(p), \ldots , f^n(p), \ldots$$

in our space from a seed "p". The behavior of orbits in higher dimensions is similar to that in one dimension. Let's look at some of the possibilities in two dimensions.

It's quite easy to devise mappings that illustrate attracting and repelling fixed points. For example,

$$f: (x, y) \dashrightarrow (x/2, y/2)$$

draws all points asymptotically towards the origin while

$$f: (x, y) \dashrightarrow (2x, 2y)$$

drives them away to infinity. The mapping

$$f: (x, y) \dashrightarrow (-y, x)$$

will set every point on its own four-cycle around the origin. Generating a mapping that produces its own unique n-cycle was a bit more difficult. One candidate

$$f: (x, y) \dashrightarrow (\cos y^{-1}, \sin y^{-1})$$

seems to send seeds on to an 11-cycle, but I'm not quite sure. This area of mathematics is highly dependent on computing and I have no computer program for generating arbitrary two-dimensional iterated mappings at my disposal (although I manage it on a programmable calculator).

In higher dimensions, however, attraction and repulsion are not limited to points. An iterative map can collapse on to any structure possible in that dimension. Attractors and repellers can form paths, surfaces, volumes, and their higher dimensional analogs. For example, the two-dimensional map

$$f: (x, y) \dashrightarrow (x, y/2)$$

attracts all points asymptotically to the x-axis. Likewise, a two-dimensional object can act as a repeller. Such is the case for the map

$$f: (x, y) \dashrightarrow (x^2, 2xy).$$

Points inside the unit circle head for the origin while those outside fly off to infinity. Points on the circle remain there and thus (for this map) the unit circle can be considered a fixed repeller.

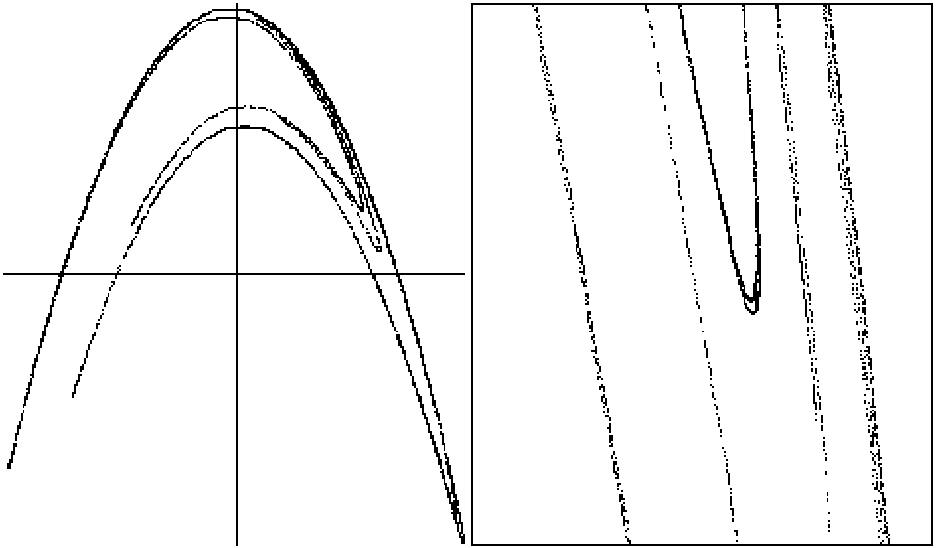
For comparison, take the set of iterated functions

$$x \dashrightarrow by \quad \text{and} \quad y \dashrightarrow 1 + x - ay^2$$

where "a" and "b" are constants set at 1.4 and 0.3 respectively. Those seed values that do not escape to infinity collapse on to the bizarre creature shown below. This is an example of a strange attractor; the Henon attractor, named after its discoverer, Michel Henon. Although composed of lines, orbits on this beast do not flow continuously, but hop from one location to another. When drawn, the Henon attractor seems to materialize out of nothing. It is also chaotic. All seed values that converge to the attractor do so in a different manner. Distinct points that are initially separated by even the most minuscule gap will eventually diverge and evolve separately. The Henon attractor also shows a great deal of fine structure (an infinite amount to be exact). Successive magnifications show an ever increasing degree of detail. Any cross-section through an arm of the Henon attractor is equivalent to a Cantor middle thirds set.

The Henon attractor with successive magnifications

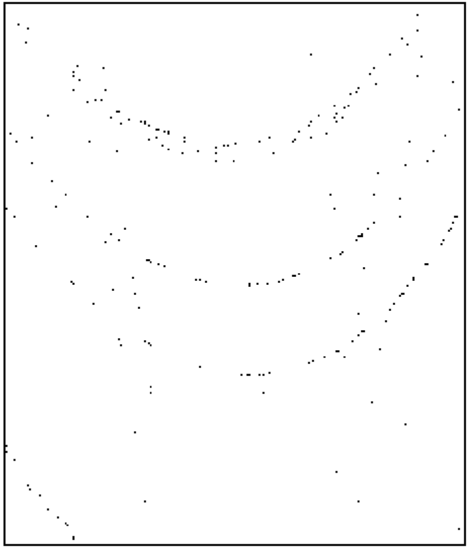
f: (x, y) --> (0.3 y, 1 + x - 1.4 y^2)



x = [-0.4046, +0.4010]
y = [-1.3490, +1.3370]

x = [0.1500, 0.2500]
y = [0.2600, 0.3600]

x = [0.2070, 0.2200]
y = [0.3020, 0.3100]

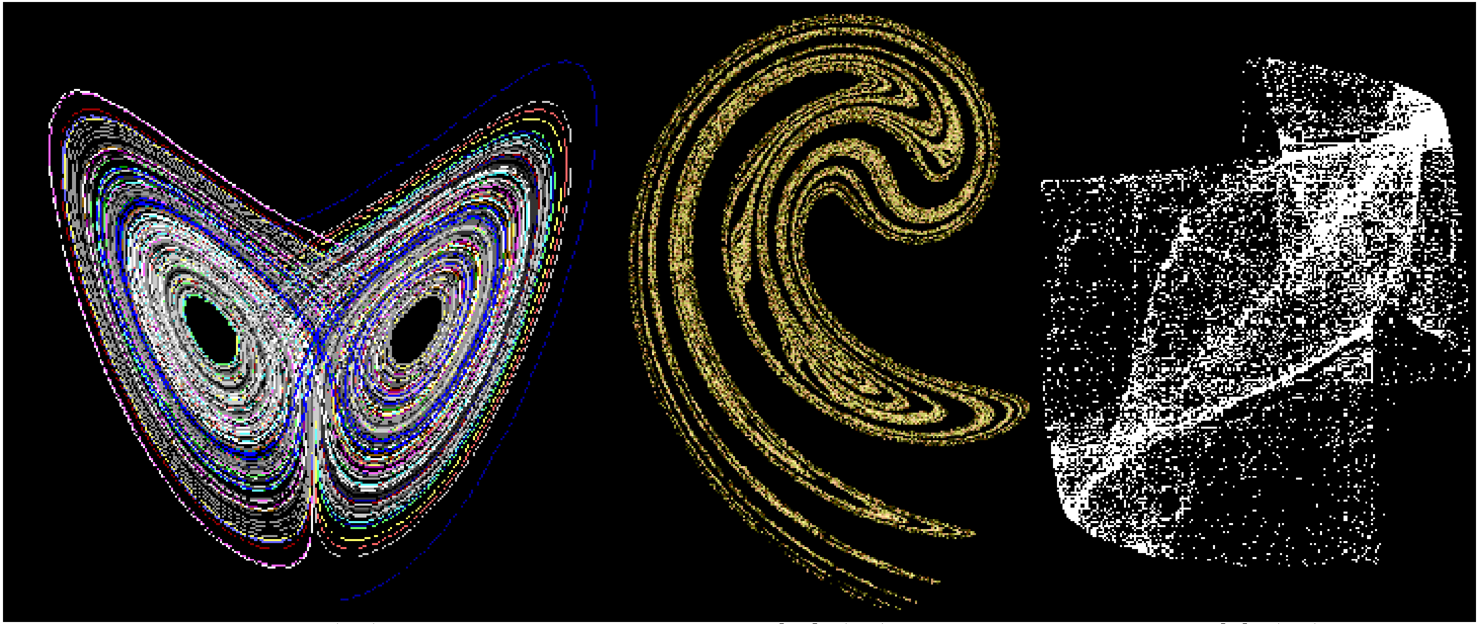


x = [0.2103, 0.2110]
y = [0.3050, 0.3055]

There are dozens of well-studied strange attractors. A few examples are shown below. The most famous is without a doubt the Lorenz attractor. This attractor emerged from the study of equations used in the prediction of weather. It was here, sometime in the early '60s, that chaos was first seen.

Some well known strange attractors

2.1 Strange Attractors



Lorenz (3D) Ikeda (2D) Ushiki (2D)

Strange & Complex

Julia sets drawn with Object Mandelbrot & Julia's Dream

2.2 Julia Sets

Let's return for a while to our original map

$$f: x \to x^2 + c.$$

The graph of this function is a parabola when "x" and "c" are real numbers. The orbits of well-behaved seeds are bounded for parameter values in the interval [-2, 1/4]. These orbits can settle on to attracting fixed points, be periodic, or ergodic. A small set of fixed points, the repelling fixed points, do not generate orbits in the traditional sense. They neither roam nor run off to infinity and one need not wait for them to exhibit "characteristic" behavior. They are permanently and immutably fixed and nearby points avoid them. They lie on the frontier between those seeds with bounded orbits and those with unbounded orbits. Such is the behavior in general for all points and all parameter values; or is it? The discussion so far has been constrained by a prejudice for real numbers. What happens when we admit that i = sqrt(-1) has a solution? How does our function behave when "z" and "c" are complex numbers? The answer, of course, is the same but the results are much more interesting than such a flip statement implies.

The map

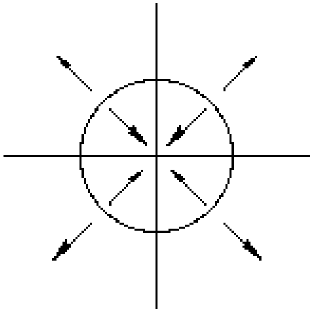
$$f: z \to z^2 + c$$

is equivalent to the two-dimensional map

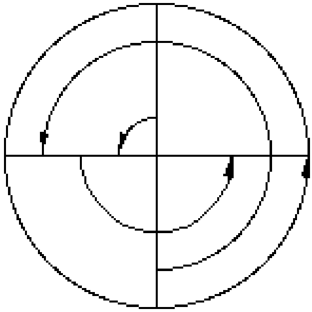
$$f: (x, y) \to (x^2 - y^2 + a, 2xy + b)$$

where $z = x + iy = (x, y)$ is a point to be iterated and $c = a + ib = (a, b)$ acts as the parameter. Thus, the quadratic map of the complex numbers can be studied as a family of transformations to the complex plane.

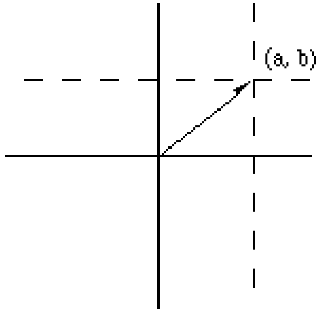
The quadratic map on the complex plane



Stretch points inside the unit circle towards the origin. Stretch points outside towards infinity



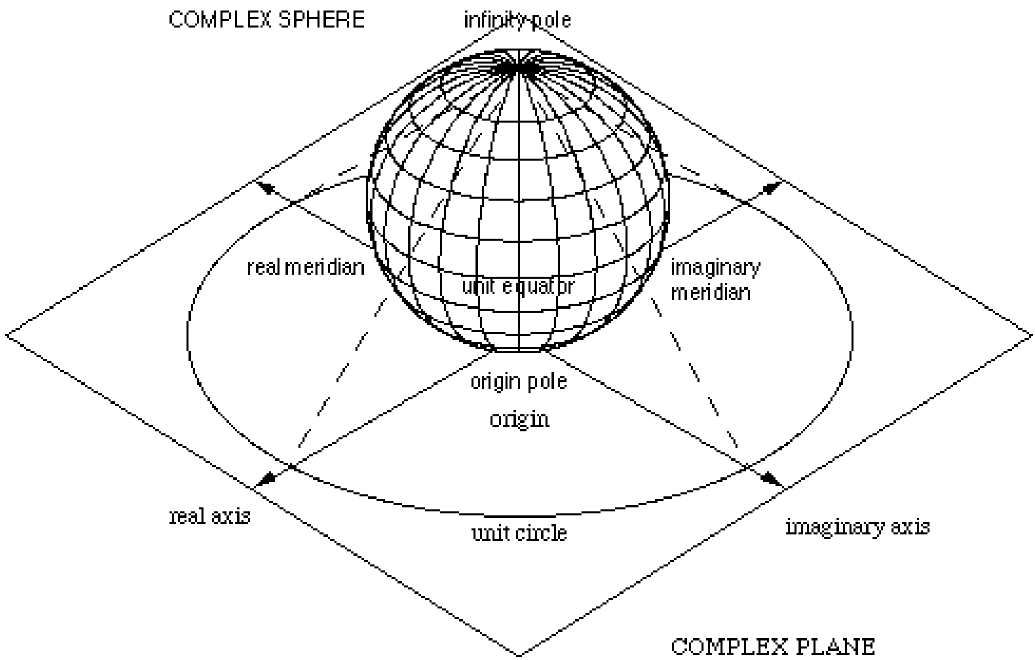
Cut along the positive x-axis. Wrap the plane around itself once by doubling every angle.



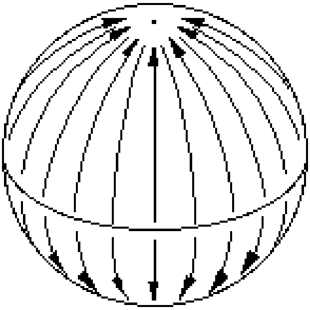
Shift the plane over so the origin lies on (a, b).

Actually, it's easier to discuss these transformations if we represent the complex numbers as points on a complex sphere. The origin would be one pole and infinity another pole with the unit sphere being the equator. Imagine placing a light bulb at the infinity pole. Points on the sphere would leave shadows in unique positions on the plane. The complex plane is thus a projection of the complex sphere. While this is easier to deal with because we have a single point called infinity, it is not easier to diagram. Let's give it a shot.

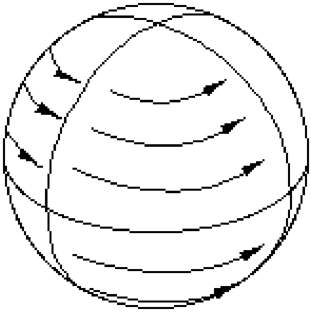
Stereographic projection of the complex sphere on to the complex plane



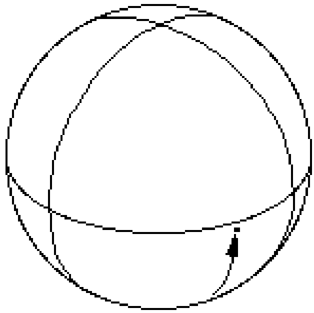
The quadratic map on the complex sphere



Stretch points below the unit equator towards the origin pole. Stretch points above the equator towards the infinity pole.



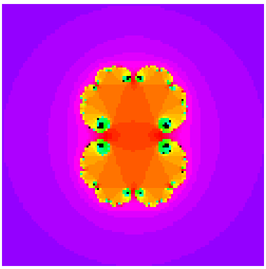
Cut along the positive real meridian Wrap the sphere around itself once by doubling every longitude value.



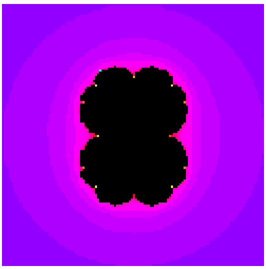
Stretch the sphere so the origin pole lies on (a, b) but the infinity pole doesn't move.

This family of mappings is said to be conformal; that is, it leaves angles unchanged. Despite all this stretching, twisting, and shifting there is always a set of points that transforms into itself. Such sets are called the Julia sets after the French mathematician Gaston Julia who first conceived of them in the 1910s. The special case of c = (0, 0) has already been dealt with (the set is the unit circle). Let's look at a variety of parameter values and see what kind of Julia sets arise. Some examples are shown below.

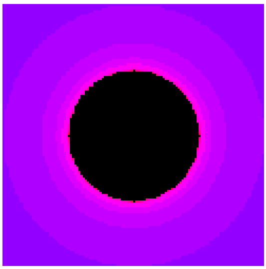
Some Julia Sets
The set is the boundary between the black and colored regions.



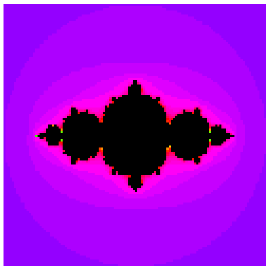
c = 0.275



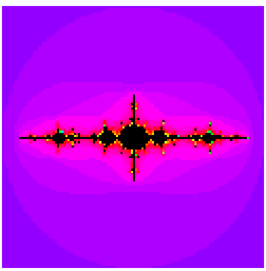
c = 1/4

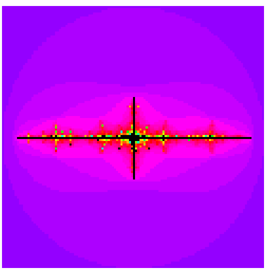


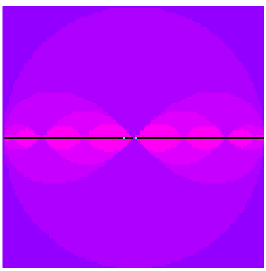
c = 0

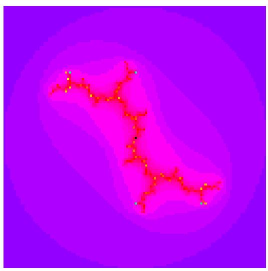


c = -3/4

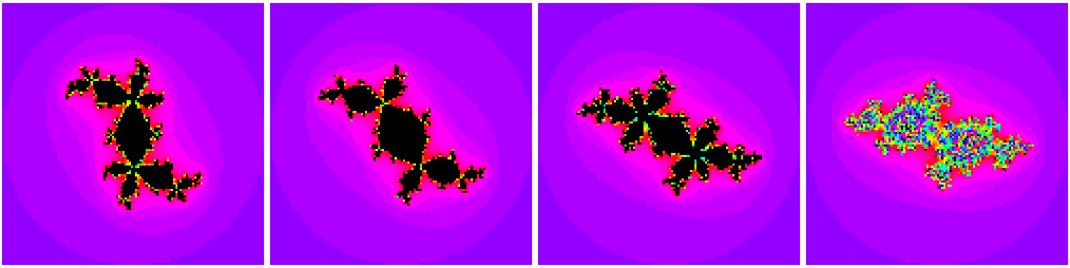








c = -1.312 | c = -1.375 | c = -2 | c = i



c = (+0.285, +0.535) | c = (-0.125, +0.750) | c = (-0.500, +0.563) | c = (-0.687, +0.312)

Sets on the real axis are reflection symmetric while those with complex parameter values show rotational symmetry. With the exception of the parameter value c = 0, all Julia sets exhibit self-similarity. There are two broad categories of Julia sets: those which are connected and those which are not. Of the twelve examples shown, only the first and last are disconnected. The distinction between the two categories is extreme. Disconnected sets are completely disconnected into a countably infinite assembly of isolated points. In addition, these points are arranged in dense groups such that any finite disk surrounding a point contains at least one other point in the set. Such sets are said to be dustlike. As they can be shown to be similar to the Cantor middle thirds set, they are often called Cantor dusts. In contrast, the connected sets are completely connected. Topologically, they are either equivalent to a severely deformed circle or to a line with an infinite series of branches and sub-branches called a dendrite (at c = i for example).

Strange & Complex

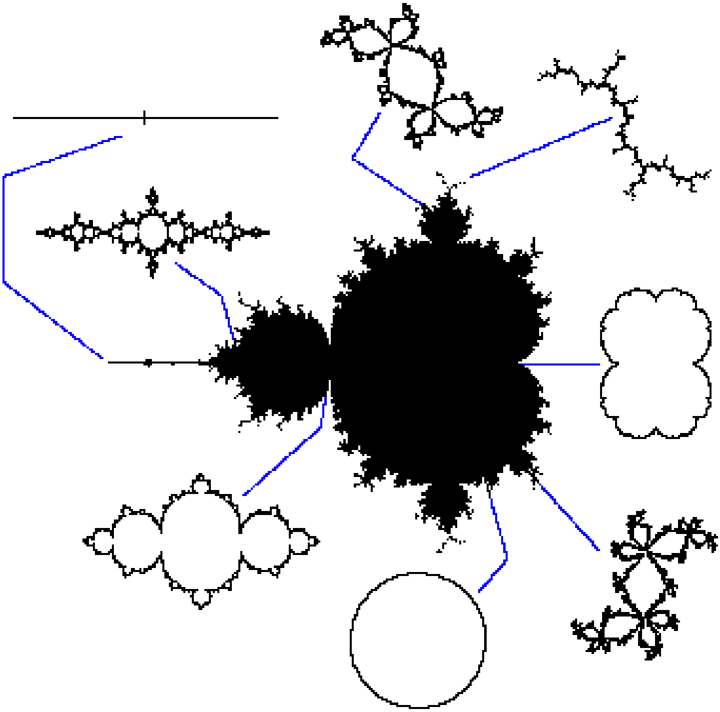
The Chaos Hypertextbook

Julia sets drawn with Julia's Dream
Mandelbrot sets drawn with MandelZot and Object Mandelbrot
Movies rendered with MandelMovie

2.3 Mandelbrot Sets

The factor that determines whether a Julia set is wholly connected or wholly disconnected is the parameter value. Thus it would be instructive to plot the behavior of the Julia sets for all parameter values. The resulting construction would be the complex analog of a bifurcation drawing. At first glance, this seems a daunting task. Plotting every possible Julia set and then examining it to determine whether it was connected or not would take an eternity. Luckily for us, however, we need only study the behavior of one point in the complex plane. Given a family of complex iterative maps, the set of all parameter values that produce wholly connected Julia sets is determined by the behavior of a single seed value: the origin. If the orbit of the origin never escapes to infinity then it is either a part of the set or it is trapped inside it. If the origin is a part of the set, the set is dendritic. If it is trapped inside the set, the set is topologically equivalent to a circle and thus is wholly connected. This trick was discovered by the Polish-American mathematician Benoit Mandelbrot and in his honor the set of all parameter values whose Julia sets are wholly connected is called a Mandelbrot set. The Mandelbrot set for the quadratic mapping f: $z \rightarrow z^2 = c$ is shown below for all parameters $c = x + iy$ in the range $x = [-2, 1/2]$ $y = [-2, 2]$. Some wholly connected Julia sets were also added and their approximate location in parameter space indicated. This type of arrangement is known as a constellation diagram.

A Mandelbrot Set with 8 accompanying Julia Sets in a constellation diagram



Note the similarities between any Julia set and the corresponding parameter space region on the Mandelbrot set. The Julia sets that are pinched vertically are found where the Mandelbrot set is pinched vertically. Likewise the set which is pinched horizontally is found on the extreme right side in a region that is pinched horizontally. The dendrite corresponds to a parameter point on a filament at the top and the long tail to the left produces a Julia set that is similarly long and tail-like. Such similarities are found at the microscopic level as well. Not only does the Mandelbrot set tell whether a corresponding Julia set is connected or disconnected, it also suggests its appearance.

An interesting way to see the variety of Julia sets is by means of a cascade.

1. Describe a "connect-the-dots" path through the Mandelbrot set
2. Draw the corresponding Julia set at each point
3. Assemble the sets into a movie and play it

The banner graphic on the home page of this book has a cascade where the "o" would be in the word "Chaos". Another two cascades can be found below using a coloring technique common to fractal images. Points inside were colored black while those outside were assigned a particular color that indicates how quickly the point escapes to infinity. This technique accentuates some of the details that might otherwise be hidden.

Julia Set cascades

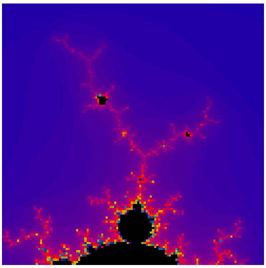
Click the icon to see the corresponding QuickTime movie

around the main bay

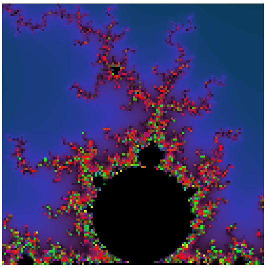
down the real axis

The main body of the Mandelbrot set is a cardioid with a series of successively smaller circles attached to it in a chain running along the x-axis in the negative direction. The attachment points on this chain correspond to the bifurcation points of the simple one-dimensional iterative map. Thus, each circle on the x-axis corresponds to a region of differing periodicity. The ratio of the diameters of successive circles approaches Feigenbaum's constant "delta". Note too, how the long tail-like region is punctuated with little islands. This corresponds to the chaotic regime and the islands to the odd period windows. Recall how in the one-dimensional case the structure of the windows was similar to the overall bifurcation diagram. Likewise these windows are miniature mutated copies of the whole Mandelbrot set. This structure also repeats itself radially around the set. Each major bulb has smaller bulbs budding off it which in turn have bulbs attached to them and so on. The whole set is bristling with filaments, each with its own array of window-like, mutated copies of the whole set. The Mandelbrot set is a wholly connected archipelago of self-similar islets linked by an array of extremely twisty, non-intersecting fibers.

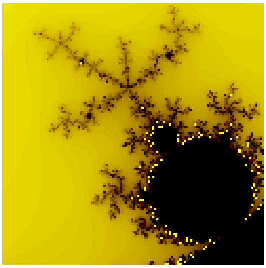
Regions of the Mandelbrot Set exhibit quasi-self-similarity
Click on an image to see a larger copy



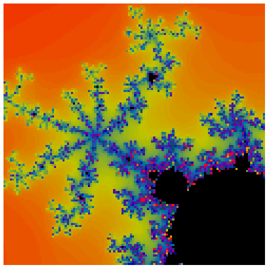
a period 3 bulb



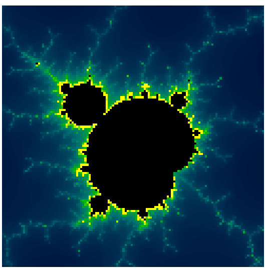
a period 4 bulb

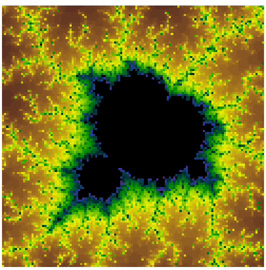


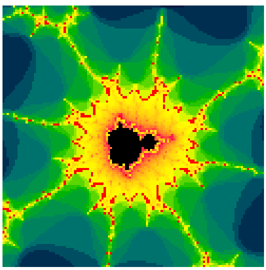
a period 5 bulb

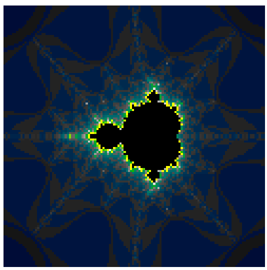


a period 7 bulb









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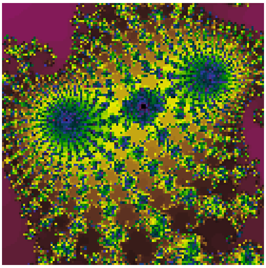
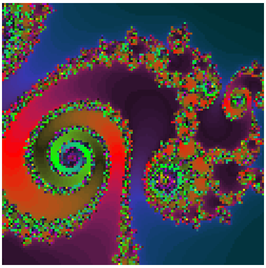
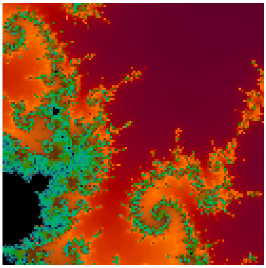
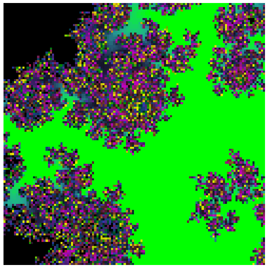
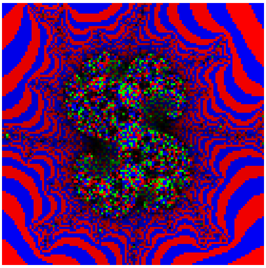
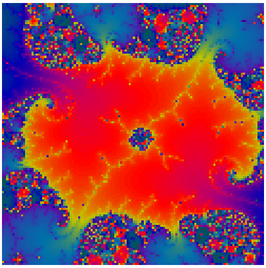
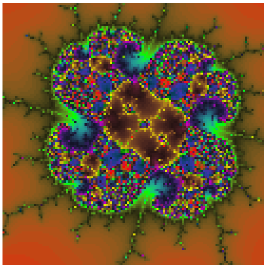
c = (-1.176..., -0.245...)
c = (-1.766..., +0.042...)
c = (-1.941..., +0.000...)

c = (-0.747..., +0.108...)
c = (-0.760..., +0.081...)
c = (-1.265..., +0.046...)
c = (+0.349..., +0.357...)


c = (-0.154..., +1.031...)
c = (-0.154..., +1.031...)
c = (-0.154..., +1.031...)
c = (-0.154..., +1.031...)
zoom in and zoom out from this image

The diagrams above give some idea of the variety of structures that can be found on the Mandelbrot set. The first four diagrams show some of the decorations attached to the bulbs surrounding the bays. Note how the branching follows a pattern unique to each bulb. Note also how the characteristic number of forks on any bulb is carried through on smaller levels of magnification. This illustrates one of the important characteristics of the Mandelbrot set. Whereas fragments of the Julia sets were strictly similar to whole set, the Mandelbrot set shows a sort of quasi-self-similarity that varies from one region to another and from one level of magnification to another.

One way to picture Mandelbrot and Julia sets is as complex ordered pairs (c, z) such that the mapping f: z --> z^2 + c does not escape to infinity when iterated. Julia sets are slices parallel to the z-axis while the Mandelbrot set is a slice along the c-axis through the origin. As the coordinate system is complex, however, these "axes" are actually planes. The Mandelbrot and Julia sets are therefore two-dimensional cross-sections through a four-dimensional parent set; the mother of all iterated quadratic mappings so to speak.

The exotic sets shown in the following movie are successive slices through the four-dimensional mother set as we shift the cross-sectional plane from the z = 0 plane of the Mandelbrot set to the c = -1 plane of a particular Julia set (the San Marco Dragon).

A Julia set transforming into a Mandelbrot set
Click the icon to load the QuickTime movie

In this chapter, I have touched on some of the topics arising out of the study of iterated mappings, expanding the simple one-dimensional case to the multi-dimensional and complex realms. By extension, one can imagine other related systems worthy of an equal amount of study. We have not addressed the analysis of other iterated mappings on the complex plane: higher powers such as $z^4 + c$ or $z^6 + c$; trigonometric functions like $\cos z + c$; hyperbolic or exponential functions, and so on. We also have not dealt with functions driven by periodic or random fluctuations, nor those with discontinuities and corners. And if we allow complex numbers, we must also allow quaternions, octernions and the rest of their higher dimensional cousins. Not surprisingly, someone has already done most of this.

What is Dimension?

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3.1 General Dimension

A space is a collection of entities called points. Both terms are undefined but their relation is important: space is superordinate while point is subordinate. Our everyday notion of a point is that it is a position or location in a space that contains all the possible locations. Since everything doesn't happen in exactly the same place, we live in what can rightly be called a space, but points need not be point-like. Any kind of object can be a point. Other geometric objects, for instance, are totally acceptable (lines, planes, circles, ellipses, conic sections) as are algebraic entities (functions, variables, parameters, coefficients) or physical measurements (time, speed, temperature, index of refraction). Even so-called "real" things can be points in a space: people are points in the space of a nation's population, nations are points in the global political space, and telephones are points in the space of a telecommunications network.

Any space that can be conceived of also has a characteristic number associated with it called a dimension. From the time I began the serious study physics and mathematics up until I discovered chaos and fractals, I had what I thought was a complete definition of dimension. My definition of dimension (which I had assumed to be *the* definition) was the number of real number parameters needed to uniquely describe all the points in a space. Thus, the real number line is one-dimensional as it only takes one real number (parameter) to describe each real number (point). Dimension is invariant so that a plane, for example, requires two parameters in rectangular (x, y) or polar (r, theta) coordinates. Other suitable examples come to mind. The set of lines in a plane is two-dimensional as describing any one of them uniquely requires two parameters: the slope and y-intercept (m, b) or the x and y-intercepts (x_0, y_0), for example. The set of all circles in a plane is three-dimensional (two for the coordinates of the center and one for the radius) and the set of all conic sections in a plane is five-dimensional (trust me).

This situation is great for physicists, and other scientists too, as it enables them to mine the wealth of mathematical knowledge about curves, surfaces, and all the rest and apply them to the rigorous study of the natural world. It takes four numbers to adequately describe the thermodynamic state of a region containing a gas: pressure, volume, temperature, and amount of material. If we limit ourselves to a fixed amount of material and assume that the gas is ideal, we can reduce the system down to two dimensions: pressure and volume (a reasonable assumption believe it or not). Thus, anything that any mathematician has ever done in a two-dimensional space could, at least in principle, be used in the study of gases. The work done by a gas as it expands isothermally turns out to be a problem in finding the area under the curve $y = a/x$. Luckily, for the physicists of the Nineteenth Century, a mathematician had already determined the solution.

What about a point? Well, a single point is easy. It takes no numbers to uniquely identify a single isolated object. If we have only one thing, we don't need to discriminate between it and anything else so a point is zero-dimensional. What about two points? This space should be one-dimensional according to our definition as describing any part of the space requires one number. The same would be true for a space of twenty points or twenty million points. Any countable set (finite or infinite) would require one parameter to adequately describe all of its points. Does this now make a countable collection of zero-dimensional objects a one-dimensional space? I wouldn't like this to be the case. A countable set of points is very different from an uncountable set (like the real numbers) and their dimensions should reflect this difference. Maybe we have an out. My original definition said that the dimension of a space is given by the number of *real* parameters it takes to identify the points. A countable set can be adequately paired with the *whole* numbers in a manner that would assign each a unique coordinate. I do not find this distinction very satisfying, however.

What is the dimension of the space of acceptable telephone numbers? Since each telephone can be uniquely located with one number it should be one-dimensional. In actuality, however, a phone number is an area code plus an exchange prefixed on to a four-digit number making the whole sequence three-dimensional. To further complicate matters, an exchange is really a pair of coordinates. Think about old movies where a character would bark into the receiver, "Operator! Give me Klondike 5-1234." If Klondike and 5 are an ordered pair then current phone numbers are really four-dimensional. Such complications further reduce the validity of the parameter counting method for determining dimension.

Another way to think of dimension is as the degrees of freedom available within the space. Our choice of directions must be orthogonal, however. Motion in a particular direction cannot be expressed as the finite combination of infinitesimal motions in the other directions. It should not be possible to move up by the finite combination of extremely small left and forward motions. If it is, then we have over-described the dimension of the space. Physical space (euclidean three-space) is three-dimensional because there are three independent directions that objects within the space can move: up/down, left/right, and forward/backward. The surface of the earth, on the other hand, is two-dimensional as we are only free to move in one of two directions: left/right and forward/backward. Any vertical motion is the result of moving in the other two directions. Under these constraints, a countable set of points is now zero-dimensional as we have zero degrees of freedom. It is not possible to move through such a space from one point to another without leaving the space.

What about an x-shaped space (╳) composed of two line segments? Locally such a space is one-dimensional everywhere except at the intersection of the two segments where it becomes two-dimensional. How should we handle such cases? Should we go with the minimum value as the true dimension? This would make the space one-dimensional, which feels more natural. If we did, however, then the union of a point and a filled square (· ■) would be zero-dimensional. This does not feel natural. The first example (╳) should be one-dimensional and the second (· ■) two-dimensional. So far, my commonsense definitions are not agreeing with my commonsense answers. I guess it's time to crack open the books and see what the real mathematicians have to say.

What is Dimension?

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3.2 Topological Dimension

By definition, the null set (∅) and only the null set shall have the dimension -1. The dimension on any other space will be defined as one greater that the dimension of the object that could be used to completely separate any part of the first space from the rest. It takes nothing to separate one part of a countable set from the rest of the set. Since nothing (∅) has dimension -1, any countable set has a dimension of 0 (-1 + 1 = 0). Likewise, a line has dimension 1 since it can be separated by a point (0 + 1 = 1), a plane has dimension 2 since it can be separated by a line (1 + 1 = 2), and a volume has dimension 3 since it can be separated by a plane (2 + 1 = 3). We have to modify this dimension a little bit, however.

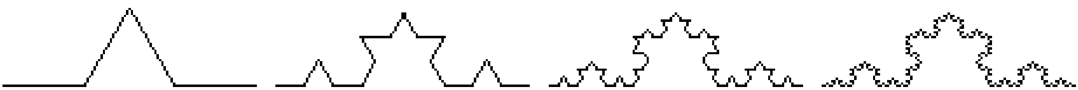
Sure a countable set can be separated by nothing, but it can also be separated by another countable set or a line or a plane. Take the rational numbers, for example. They form a countable infinite set. By embedding the set in the real number line, we could separate one point from any other with an irrational number. This set is has dimension 0, which would give the rational numbers a dimension of 1 (0 + 1 = 1). By embedding the set in the coordinate plane, we could also use any line with an x-intercept. This would give the rational numbers a dimension of 2 (1 + 1 = 2). We could also use planes if we embedded the set in a euclidean three-space and so on. I think it would be all right if we used the minimum value and called it the dimension of the space.

What about our composite spaces () and ()? We want the first to have dimension 1 and the second dimension 2. The x-shaped space is no problem. The least dimensional entity needed to separate it would be a point even at the intersection. The point and filled square is a bit more challenging. We need to distinguish between local dimension and global dimension. If we use the last definition and apply it to the set as a whole, then the space () would have dimension 0. If on the other hand, we examine it region by region we find that the point part has dimension 0 while any part of the square region has dimension 2. This is an example of a local dimension. The global dimension of the whole space should be two-dimensional so we need to modify our definition slightly. The dimension of a space should be the maximum of its local dimensions where the local dimension is defined as one more than the dimension of the lowest dimensional object with the capacity to separate any neighborhood of the space into two parts.

The measure defined above is called the topological dimension of a space. A topological property of an entity is one that remains invariant under continuous, one-to-one transformations or homotopies. A homotopy can best be envisioned as the smooth deformation of one space into another without tearing, puncturing, or welding it. Throughout such processes, the topological dimension does not change. A sphere is topologically equivalent to a cube since one can be deformed into the other in such a manner. Similarly, a line segment can be pinched and stretched repeatedly until it has lost all its straightness, but it will still have a topological dimension of 1. Take the example below.

1. Start with a line segment. Divide it into thirds. Place the vertex of an equilateral triangle in the middle third.
2. Copy the whole curve and reduce it to 1/3 its original size. Place these reduced curves in place of the sides of the original.
3. Repeat step 2.

The result the is so-called Koch coastline, which evolves something like this.



With each iteration the curve length increases by the factor 4/3. The infinite repeat of this procedure sends the length off to infinity. The area under the curve, on the other hand, is given by the series

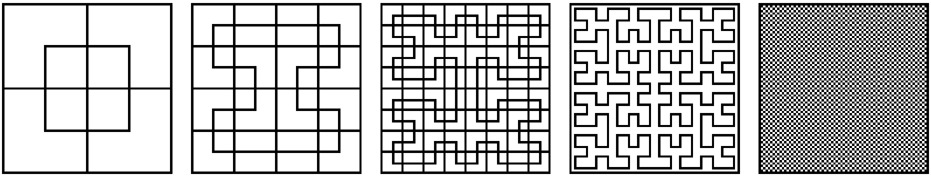
$$1 + (4/9) + (4/9)^2 + (4/9)^3 + \ldots$$

which converges to 9/3 (assuming the area under the first curve is 1). These results are unusual but not disturbing. Such is not the case for the next curve.

1. Take a unit square. Call it a cell.
2. Divide each cell into four identical miniature copies of the original cell

3. Draw a line starting in one cell so that it passes through every other cell until it returns to the starting position. (Also make sure the line does not stray too far from the previous iteration of the curve.)
4. Repeat step 2

The result is something like the diagram below. (Cell lines were omitted in the third iteration for clarity. The last diagram represents the hypothetical result of an infinite iteration.)



This curve twists so much that it has infinite length. More remarkable is that it will ultimately visit every point in the unit square. Thus, there exists a one-to-one mapping from the points on a line segment to the points in the unit plane. In other words, an object with topological dimension 1 can be transformed into an object with topological dimension 2 through a procedure that should not allow for such an occurrence. Simple bending and stretching should leave the topological dimension unchanged, however. This is a Peano monster curve (Hilbert's version), so called because of its monstrous or pathological nature. According to one source (Kline), we really have nothing to fear from the monster. The mapping from the line to the plane may be one-to-one but it is not continuous and is thus not a homotopy. This explains the change in dimension, but I am not satisfied. Kline never explains why the mapping is not continuous. I have never been happy with textbooks that leave the proof as an exercise for the reader.

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Data calculated using Fractal Dimension Calculator
San Marco dragon drawn with Julia's Dream

3.3 Fractal Dimension

There really was a reason to fear pathological entities like the Koch coastline and Peano's monster curve. Here were creations so twisted and distorted that they did not fit into the box of contemporary mathematics. Luckily, mathematics was fortified by the study of the monsters and not destroyed by them. Whatever doesn't kill you only makes you stronger.

Take the Koch coastline and examine it through a badly focused lens. It appears to have a certain length. Let's call it 1 unit. Sharpen the focus a bit so that you can resolve details that are 1/3 as big as those seen with the first approximation. The curve is now four times longer or 4 units. Double the resolution by the same factor. Using a focus that reveals details 1/9 the first focus gives us a coastline 16 times longer and so on. Such an activity hints at the existence of a quantifiable characteristic.

To be a bit more precise, every space that feels "real" has associated with it a sense of distance between any two points. On a line segment like the Koch coastline, we arbitrarily chose the length of one side of the first iterate as a unit length. On the Euclidean coordinate plane the distance between any two points is given by the Pythagorean theorem

$$s^2 = x^2 + y^2.$$

In relativity, the "distance" between any two events in space-time is given by the proper time

$$s^2 = c^2t^2 - x^2 - y^2 - z^2.$$

Such distance establishing relationships are called metrics and a space that has a metric associated with it is called a metric space. One of the more famous, non-euclidean metrics is the Manhattan metric (or taxicab metric). How far is the corner of 33rd and 1st from 69th and 5th? Answer: 36 blocks and 4 avenues or 40 units. (We have to bend reality a bit and assume that city blocks in Manhattan are square and not rectangular.) Metrics are also used to create neighborhoods in a space. Pick a point in a metric space. This point plus all others lying less than or equal to a certain distance away comprise a region of the space called a closed disk. The term disk is used because such regions are disk-shaped in the coordinate plane with the usual metric but any shape is possible. In euclidean three-space disks would be balls while in a two-space with a Manhattan metric they would be squares.

How many disks does it take to cover the Koch coastline? Well, it depends on their size of course. 1 disk with diameter 1 is sufficient to cover the whole thing, 4 disks with diameter 1/3, 16 disks with diameter 1/9, 64 disks with diameter 1/27, and so on. In general, it takes 4^n disks of radius $(1/3)^n$ to cover the Koch coastline. If we apply this procedure to any entity in any metric space we can define a quantity that is the equivalent of a dimension. The Hausdorff-Besicovitch dimension of an object in a metric space is given by the formula

$$D = \lim_{h \to 0} \frac{\log N(h)}{\log(\frac{1}{h})}$$

where N(h) is the number of disks of radius h needed to cover the object. Thus the Koch coastline has a Hausdorff-Besicovitch dimension which is the limit of the sequence

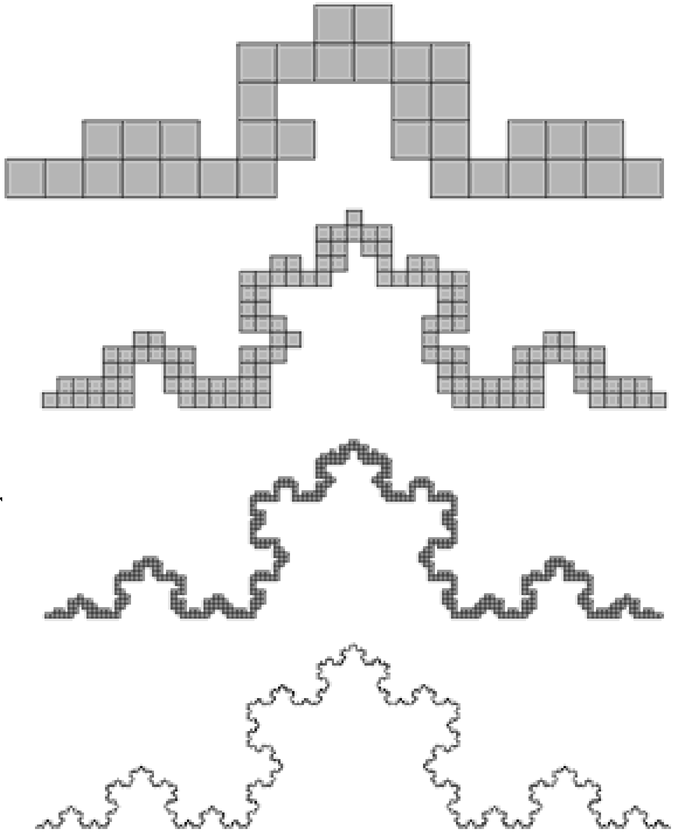
$$\frac{\log 1}{\log 1}, \frac{\log 4}{\log 3}, \frac{\log 16}{\log 9}, \frac{\log 64}{\log 27}, \ldots \frac{\log 4^n}{\log 3^n} = \frac{n \log 4}{n \log 3} = \frac{\log 4}{\log 3} = 1.261859507\ldots$$

Is this really a dimension? Apply the procedure to the unit line segment. It takes 1 disk of diameter 1, 2 disks of diameter 1/2, 4 disks of diameter 1/4, and so on to cover the unit line segment. In the limit we find a dimension of

$$\frac{\log 2^n}{\log 2^n} = \frac{n \log 2}{n \log 2} = \frac{\log 2}{\log 2} = 1$$

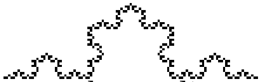
This agrees with the topological dimension of the space.

The problem now is, how do we interpret a result like 1.261859507...? This does not agree with the topological dimension of 1 but neither is it 2. The Koch coastline is somewhere between a line and a plane. Its dimension is not a whole number but a fraction. It is a fractal. Actually fractals can have whole number dimensions so this is a bit of a misnomer. A better definition is that a fractal is any entity whose Hausdorff-Besicovitch dimension strictly exceeds its topological dimension ($D > D_T$). Thus, the Peano space-filling curve is also a fractal as we would expect it to be. Even though its Hausdorff-Besicovitch dimension is a whole number ($D = 2$) its topological dimension ($D_T = 1$) is strictly less than this. The monster has been tamed.



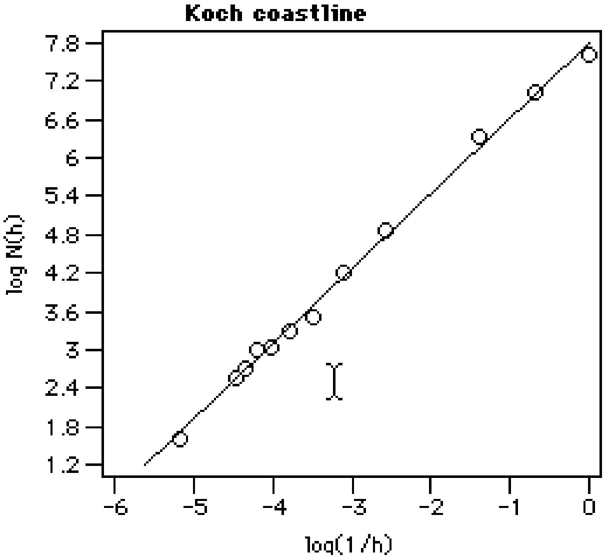
It should be possible to use analytic methods like those described above on all sorts of fractal objects. Whether this is convenient or simple is another matter. Fractals produced by simple iterative scaling procedures like the Koch coastline are very easy to handle analytically. Julia and Mandelbrot sets, fractals produced by the iterated mapping of continuous complex functions, are another matter. There's no obvious fractal structure to the quadratic mapping, no hint that a "monster" curve lurks inside, and no simple way to extract an exact fractal dimension. If there are analytic techniques for calculating the fractal dimension of an arbitrary Julia set they are well hidden. A narrow and quick search of the popular literature reveals nothing on the ease or impossibility of this task. There are, however, experimental techniques.

Take any planar geometric object of finite extent (fractal or otherwise) and cover it with a single closed disk. Any type of disk will do, so to make life easy we will use a square; the disk of the Manhattan metric in the plane. Record its dimension and call it "h". Repeat the procedure with a smaller box. Record its dimension and the number of boxes "N(h)" required to cover the object. Repeat with ever smaller boxes until you have reached the limit of your resolving power as shown in the figure to the right. Plot the results on a graph with "log N(h)" on the vertical axis and "log (1/h)" on the horizontal axis. The slope of the best fit line of the data will be an approximation of the Hausdorff-Besicovitch dimension of the object. The following are the results of a few sample experiments using this box-counting method. I think with a bit of refinement, the deviations could all be brought below 5%.

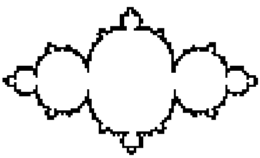


Koch coastline

log (1/h)	log N(h)
0	7.60837
-0.693147	7.04054
-1.38629	6.32972
-2.56495	4.85981
-3.09104	4.21951
-3.49651	3.52636
-3.78419	3.29584
-4.00733	3.04452
-4.18965	2.99573
-4.34381	2.70805
-4.47734	2.56495
-5.17615	1.60944

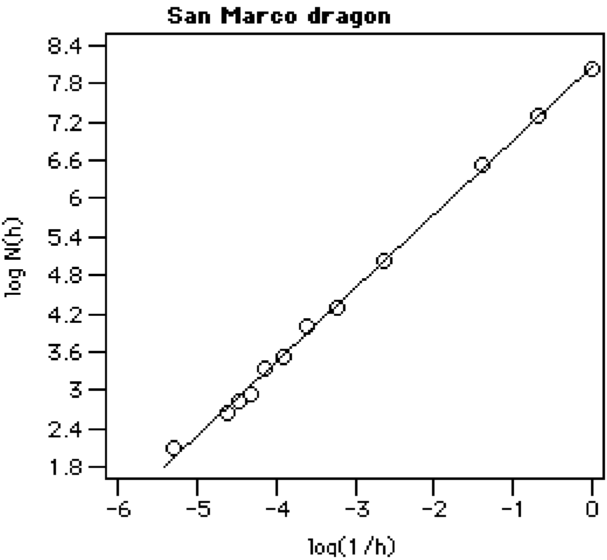


dimension (experimental) = 1.18
dimension (analytical) = 1.26
deviation = 6.35%



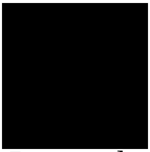
San Marco dragon

log (1/h)	log N(h)
0	8.02355
-0.693147	7.29438
-1.38629	6.52209
-2.63906	5.03044
-3.21888	4.29046
-3.61092	4.00733
-3.91202	3.52636
-4.12713	3.3322
-4.31749	2.94444



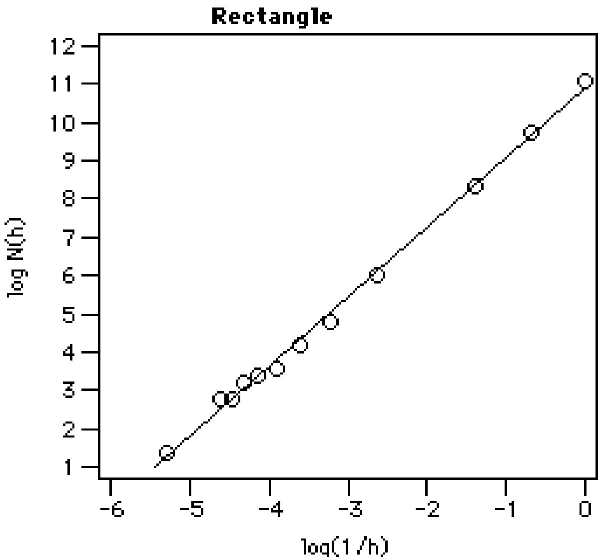
-4.46591	2.83321
-4.60517	2.63906
-5.29832	2.07944

dimension (experimental) = 1.16
dimension (analytical) = ???
deviation = ???



Rectangle

log (1/h)	log N(h)
0	11.0904
-0.693147	9.71962
-1.38629	8.31777
-2.63906	5.99146
-3.21888	4.79579
-3.61092	4.15888
-3.91202	3.58352
-4.12713	3.4012
-4.31749	3.21888
-4.46591	2.77259
-4.60517	2.77259
-5.29832	1.38629



dimension (experimental) = 1.82
dimension (analytical) = 2.00
deviation = 9.00%

Although this chapter is ending, this is not the last word on dimension. According to one source (Harrison) there are at least twelve definitions of dimension in all and another source (Hocking & Young) cites an entire book on the subject. The topics of chaos, fractals, and dimension are rich and strange. They are immensely interesting and serious consideration should be given to incorporating them into high school mathematics. I can easily envision these three topics as the central themes of a final year high school math course incorporating the basics, advanced topics, current events in science, and computer applications.

The chapter following this is intended for those with an intermediate-level college mathematics background. Of course everyone can scan through it, look at the pictures, and run the movies.

Measuring Chaos

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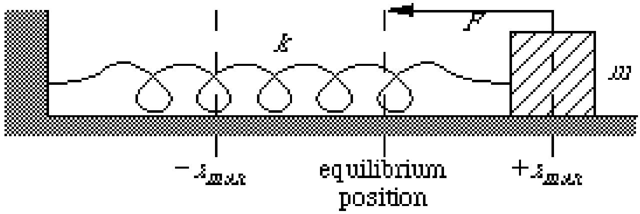
4.1 Harmonic Oscillator

A large portion of chapters 1 and 2 dealt with the behavior of discrete dynamical systems, in particular, the family of quadratic mappings. Three basic types of behavior were observed: fixed, periodic, and chaotic (or ergodic). The first two kinds of behavior are present in many continuous dynamical systems of the kind described by differential equations with exact solutions. In this chapter, a comparison will be made between the best known continuous periodic system, the harmonic oscillator, and the best known discrete periodic system, the logistic equation. Given the analogous nature of the two systems, it should be possible to perform a parallel analysis of the simple harmonic oscillator and the logistic equation. From such a study, a new topic in discrete dynamical analysis will arise, namely, an analog to the driven harmonic oscillator: what I call the driven logistic equation. In the process, a device for quantifying the behavior of iterated mappings will be introduced, the Lyapunov exponent, with the indirect result of producing ever more fascinating images.

The simple harmonic oscillator (SHO) is a mass connected to some elastic object of negligible mass that is fixed at the other end and constrained so that it may only move in one dimension. This simplified model approximates many systems that vibrate or oscillate: drum heads, guitar strings, the quantum mechanical descriptions of an atom, *etc*. The importance of this problem, however, lies in the fact that equations of a similar form arise when a particle moves through any region whose potential has one or more local minima: planetary and satellite motion, the classical description of an electron in orbit around a nucleus, pendulums, etc. Similar equations also arise in the study of LCR circuits: the type used in analog communications devices and electric power transmission. (The letters L, C, and R refer to the symbols used to identify the electrical quantities of inductance, capacitance, and resistance respectively.)

When dissipative forces such as friction and air resistance are ignored, the net force will be directly proportional to the displacement of the mass from the system's equilibrium position and pointing in the opposite direction; a condition known as Hooke's law. Beginning with Newton's second law of motion, we can derive a second order linear differential equation whose solution gives us the displacement of the mass as a function of time.

The simple harmonic oscillator



$$F_{net} = ma$$
$$-kx = m\ddot{x}$$
$$m\ddot{x} + kx = 0$$
$$x = A\cos(\omega_0 t + \phi)$$
$$\omega_0 = \sqrt{\frac{k}{m}}$$

x = displacement
k = spring constant
m = mass
A = amplitude
ϕ = phase
ω_0 = natural frequency

The motion is periodic with a frequency that depends on the nature of the mass and the elastic object (here assumed to be a spring). Amplitude (A) and phase (phi) are constants determined by the initial displacement and velocity of the system.

A more realistic physical model is one that includes dissipative forces: the damped harmonic oscillator. For the sake of simplicity, assume that any dissipative force is directly proportional to the velocity of the mass and in the opposite direction. This is a good approximation of the behavior of air resistance and produces another differential equation with an exact solution. In fact, it is the only type of dissipative force for which the differential equation of motion has an exact solution.

The damped harmonic oscillator

$$F_{net} = ma$$
$$-b\dot{x} - kx = m\ddot{x}$$
$$m\ddot{x} + b\dot{x} + kx = 0$$
$$x = Ae^{-\gamma t}\cos(\omega_1 t + \phi)$$
$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \gamma = \frac{b}{2m} \qquad \omega_1 = \sqrt{\omega_0^2 - \gamma^2}$$

b = coefficient of drag
γ = damping factor
ω_1 = damped frequency

We now have an equation that yields different behavior for different parameter values. When the damping factor equals zero the system reduces to the case of the simple harmonic oscillator: continuous oscillation at the natural frequency with constant amplitude. When the damping factor is greater than zero the system may or may not oscillate, depending of the relation between the damping factor (gamma) and the damped frequency (omega$_0$).

omega$_0$ > gamma — The system is said to be under damped and exhibits transient behavior, oscillating at the damped frequency with an amplitude that decays exponentially. If we wait long enough the system will settle into its equilibrium position.

omega$_0$ = gamma — The critically damped case. The system will return quickly and smoothly to its equilibrium position. There is no oscillatory behavior at all this time. The motion is described entirely by exponential decay.

omega$_0$ < gamma — The over damped case. The solution is now the sum of two exponential decay terms, one slower than the other, and is of the form

$$x = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}$$

where $\gamma_1 = \gamma + \sqrt{\gamma^2 - \omega_0^2}$ & $\gamma_2 = \gamma - \sqrt{\gamma^2 - \omega_0^2}$.

The motion approaches a steady state, but more slowly than in the critically damped case.

Another common mechanical problem arises when a damped harmonic oscillator is driven by some time-dependent external applied force: the driven harmonic oscillator. The most important case is that of a force that oscillates in a sinusoidal manner. If the driving force is of the form

$$F(t) = F_0 \cos(\omega t + \phi_0)$$

then the differential equation has an exact solution.

The driven harmonic oscillator

$$F_{net} = ma \qquad F_0 = \text{maximum driving force}$$
$$F(t) - b\dot{x} - kx = m\ddot{x} \qquad \phi_0 = \text{driving phase}$$
$$F(t) = F_0\cos(\omega t + \phi_0) \qquad \omega = \text{driving frequency}$$
$$m\ddot{x} + b\dot{x} + kx = F_0\cos(\omega t + \phi_0)$$

$$x = Ae^{-\gamma t}\cos(\omega_1 t + \phi) + \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}\sin(\omega t + \phi_0 + \beta)$$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \gamma = \frac{b}{2m} \qquad \omega_1 = \sqrt{\omega_0^2 - \gamma^2} \qquad \beta = \tan^{-1}\frac{\omega_0^2 - \omega^2}{2\gamma\omega}$$

The solution has two parts: transient and steady state. The transient portion, which has the same solution as the damped harmonic oscillator, dies out exponentially and depends on the initial conditions. The steady state portion has an amplitude that remains constant and does not depend on the initial conditions. Thus no mater what initial conditions the oscillator had, it will eventually acquire behavior that is wholly dependent upon the driving force.

The amplitude that the oscillator eventually acquires depends on the relation of the driving frequency to the natural frequency of the oscillator and on the damping factor. It is a maximum when

$$(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2$$

is a minimum. This occurs when the ratio of the two frequencies is equal to

$$\sqrt{1+\gamma^2}\,.$$

This condition is known as resonance and results in a large amplitude of oscillation. When the driving frequency equals the natural frequency, the amplitude of the steady state portion of the solution

$$\frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \quad \text{reduces to} \quad \frac{F_0/m}{2\gamma\omega}.$$

As the damping factor approaches zero, the steady state amplitude approaches infinity. This illustrates the importance of damping in structures susceptible to vibration such as suspension bridges and steel framed buildings. Without damping, a structure could shake itself to pieces from a tiny external force with just the right frequency.

A classic demonstration of resonance is the Tacoma Narrows bridge incident. Long span bridges are now all designed with open cross section struts to dissipate some of the wind's force. The former Paramount Communications building on Columbus Circle recently converted by Donald Trump into luxury apartments provides a more contemporary example. The base of the building, which is a long narrow prism, was wrapped in scaffolding for the better part of five years. Because of a structural flaw that allowed the top floors to twist, windows were prying loose from their casements. A degree of flexibility is needed in large buildings, but it was feared that gale force winds would pop the windows out violently, showering the streets below with plate glass. Part of the building's renovation included structural repairs to dampen this motion.

In summary, we have seen how a second order linear differential equation, the simple harmonic oscillator, can generate a variety of behaviors. In the damped harmonic oscillator we saw exponential decay to an equilibrium position with natural periodicity as a limiting case. The determining factor that described the system was the relation between the natural frequency and the damping factor. In the driven harmonic oscillator we saw transience leading to some steady state periodicity. The final behavior of the system depended on the relation between the driving frequency and the natural frequency (and to a lesser extent the damping factor). The behaviors described above are also found in first order nonlinear difference equations; the quadratic mapping and the related logistic equation. I will review the latter of these and present it in a manner similar to what has appeared so far.

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Bifurcation diagram drawn with 1-D Chaos Explorer

4.2 Logistic Equation

The simple logistic equation is a formula for approximating the evolution of an animal population over time. Many animal species are fertile only for a brief period during the year and the young are born in a particular season so that by the time they are ready to eat solid food it will be plentiful. For this reason, the system might be better described by a discrete difference equation than a continuous differential equation. Since not every existing animal will reproduce (a portion of them are male after all), not every female will be fertile, not every conception will be successful, and not every pregnancy will be successfully carried to term; the population increase will be some fraction of the present population. Therefore, if "A_n" is the number of animals this year and "A_{n+1}" is the number next year, then

$$A_{n+1} = rA_n$$

where "r" is the growth rate or fecundity, will approximate the evolution of the population. This model produces exponential growth without limit. Since every population is bound by the physical limitations of its surrounding, some allowance must be made to restrict this growth. If there is a carrying-capacity of the environment then the population may not exceed that capacity. If it does, the population would become extinct. This can be modeled by multiplying the population by a number that approaches zero as the population approaches its limit. If we normalize the "A_n" to this capacity then the multiplier $(1 - A_n)$ will suffice and the resulting logistic equation becomes

$$A_{n+1} = rA_n(1 - A_n)$$

or in functional form

$$f(x) = rx\,(1 - x).$$

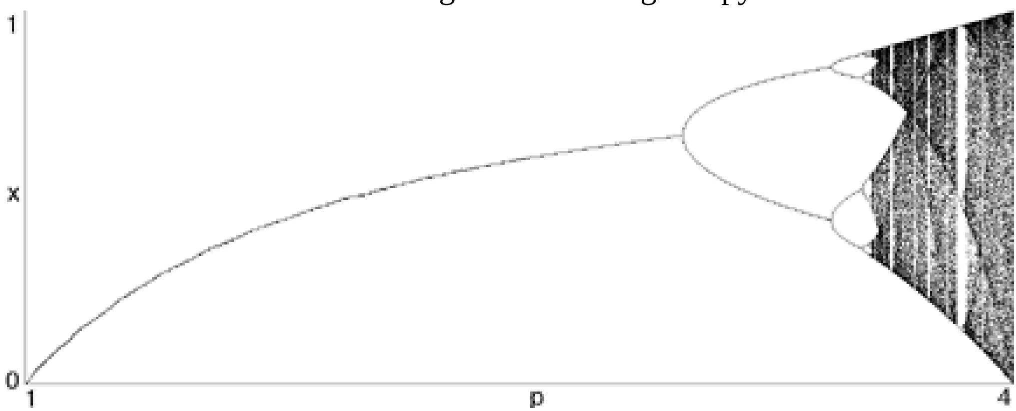
The logistic equation is parabolic like the quadratic mapping with $f(0) = f(1) = 0$ and a maximum of 1/4 r at 1/2. Varying the parameter changes the height of the parabola but leaves the width unchanged. (This is different from the quadratic mapping which kept its overall shape and shifted up or down.) The behavior of the system is determined by following the orbit of the initial seed value. All initial conditions eventually settle into one of three different types of behavior.

1. **Fixed**: The population approaches a stable value. It can do so by approaching asymptotically from one side in a manner something like an over damped harmonic oscillator or asymptotically from both sides like an under damped oscillator. Starting on a seed that is a fixed point is something like starting an SHO at equilibrium with a velocity of zero. The logistic equation differs from the SHO in the existence of eventually fixed points. It's impossible for an SHO to arrive at its equilibrium position in a finite amount of time (although it will get arbitrarily close to it).
2. **Periodic**: The population alternates between two or more fixed values. Likewise, it can do so by approaching asymptotically in one direction or from opposite sides in an alternating manner. The nature of periodicity is richer in the logistic equation than the SHO. For one thing, periodic orbits can be either stable or unstable. An SHO would never settle in to a periodic state unless driven there. In the case of the damped oscillator, the system was leaving the periodic state for the comfort of equilibrium. Second, a periodic state with multiple maxima and/or minima can arise only from systems of coupled SHOs (connected or compound pendulums, for example, or vibrations in continuous media). Lastly, the periodicity is discrete; that is, there are no intermediate values.
3. **Chaotic**: The population will eventually visit every neighborhood in a subinterval of (0, 1). Nested among the points it does visit, there is an uncountable set of fixed points and periodic points of every period. The points are equivalent to the Cantor middle thirds set and are wildly unstable. It is highly likely that any real population would ever begin with one of these values. In addition, chaotic orbits exhibit sensitive dependence on initial conditions such that any two nearby points will eventually diverge in their orbits to any arbitrary separation one chooses.

The behavior of the logistic equation is more complex than that of the simple harmonic oscillator. The type of orbit depends on the growth rate parameter, but in a manner that does not lend itself to "less than", "greater than", "equal to" statements. The best way to visualize the behavior of the orbits as a function of the growth rate is with a bifurcation diagram. Pick a convenient seed value, generate a large number of iterations, discard the first few and plot the rest as a function of the growth factor. For parameter values where the orbit is fixed, the bifurcation diagram will reduce to a single line; for periodic values, a series of lines; and for chaotic values, a gray wash of dots.

Since the first two chapters of this work were filled will bifurcation diagrams and commentary on them, I won't go much into the structure of the diagram other than to locate the most prominent features. There are two fixed points for this function: 0 and 1 - 1/r, the former being stable on the interval (-1, +1) and the latter on (1, 3). A stable 2-cycle begins at r = 3 followed by a stable 4-cycle at r = 1 + sqrt(6). The period continues doubling over ever shorter intervals until around r = 3.5699457... where the chaotic regime takes over. Within the chaotic regime there are interspersed various windows with periods other than powers of 2, most notably a large 3-cycle window beginning at r = 1 + sqrt(8). When the growth rate exceeds 4, all orbits zoom to infinity and the modeling aspects of this function become useless.

Bifurcation diagram of the logistic equation
Click the image to view a larger copy



Measuring Chaos

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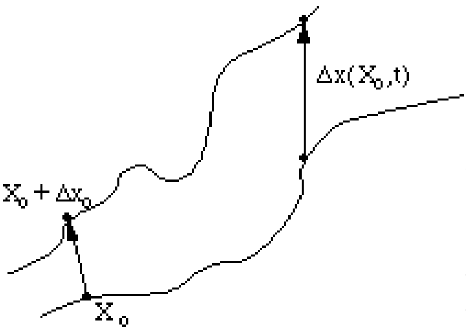
Standard map orbits drawn with Std Map

4.3 Lyapunov Exponent

Descriptions of the sort given in the last paragraph are unnatural and clumsy. It would be nice to have a simple measure that could discriminate among the types of orbits in the same manner as the parameters of the harmonic oscillator.



Consider two points in a space

$$X_0 \quad \& \quad X_0 + \text{delta-}x_0$$

each of which will generate an orbit in that space using some equation or system of equations. These orbits can be thought of as parametric functions of a variable that is something like time. If we use one of the orbits a reference orbit, then the separation between the two orbits will also be a function of time. Because sensitive dependence can arise only in some portions of a system (like the logistic equation), this separation is also a function of the location of the initial value and has the form delta-x(X_0, t). In a system with attracting fixed points or attracting periodic points, delta-x(X_0, t) diminishes asymptotically with time. If a system is unstable, like pins balanced on their points, then the orbits diverge exponentially for a while, but eventually settle down. For chaotic points, the function delta-x(X_0, t) will behave erratically. It is thus useful to study the mean exponential rate of divergence of two initially close orbits using the formula

$$\lambda = \lim_{\substack{t \to \infty \\ |\Delta x_0| \to 0}} \frac{1}{t} \ln \frac{|\Delta x(X_0, t)|}{|\Delta x_0|}$$

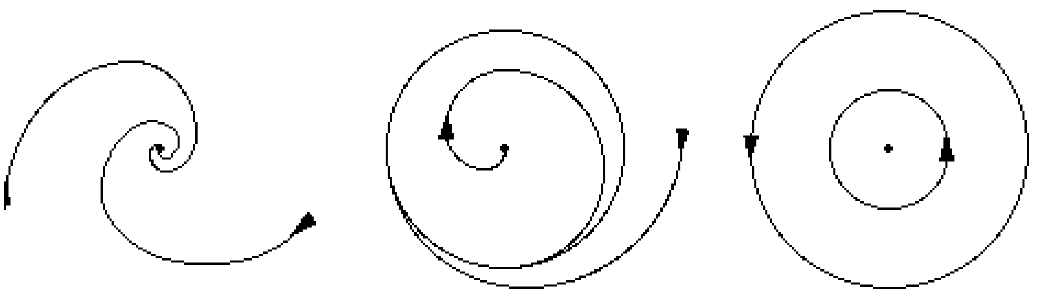
This number, called the Lyapunov exponent, is useful for distinguishing among the various types of orbits. It works for discrete as well as continuous systems.

lambda < 0 The orbit attracts to a stable fixed point or stable periodic orbit. Negative Lyapunov exponents are characteristic of dissipative or non-conservative systems (the damped harmonic oscillator for instance). Such systems exhibit asymptotic stability; the more negative the exponent, the greater the stability. Superstable fixed points and superstable periodic points have a Lyapunov exponent of $\lambda = -\infty$. This is something akin to a critically damped oscillator in that the system heads towards its equilibrium point as quickly as possible.

lambda = 0 The orbit is a neutral fixed point (or an eventually fixed point). A Lyapunov exponent of zero indicates that the system is in some sort of steady state mode. A physical system with this exponent is conservative. Such systems exhibit Lyapunov stability. Take the case of two identical simple harmonic oscillators with different amplitudes. Because the frequency is independent of the amplitude, a phase portrait of the two oscillators would be a pair of concentric circles. The orbits in this situation would maintain a constant separation, like two flecks of dust fixed in place on a rotating record.

lambda > 0 The orbit is unstable and chaotic. Nearby points, no matter how close, will diverge to any arbitrary separation. All neighborhoods in the phase space will eventually be visited. These points are said to be unstable. For a discrete system, the orbits will look like snow on a television set. This does not preclude any organization as a pattern may emerge. Thus the snow may be a bit lumpy. For a continuous system, the phase space would be a tangled sea of wavy lines like a pot of spaghetti. A physical example can be found in Brownian motion. Although the system is deterministic, there is no order to the orbit that ensues.

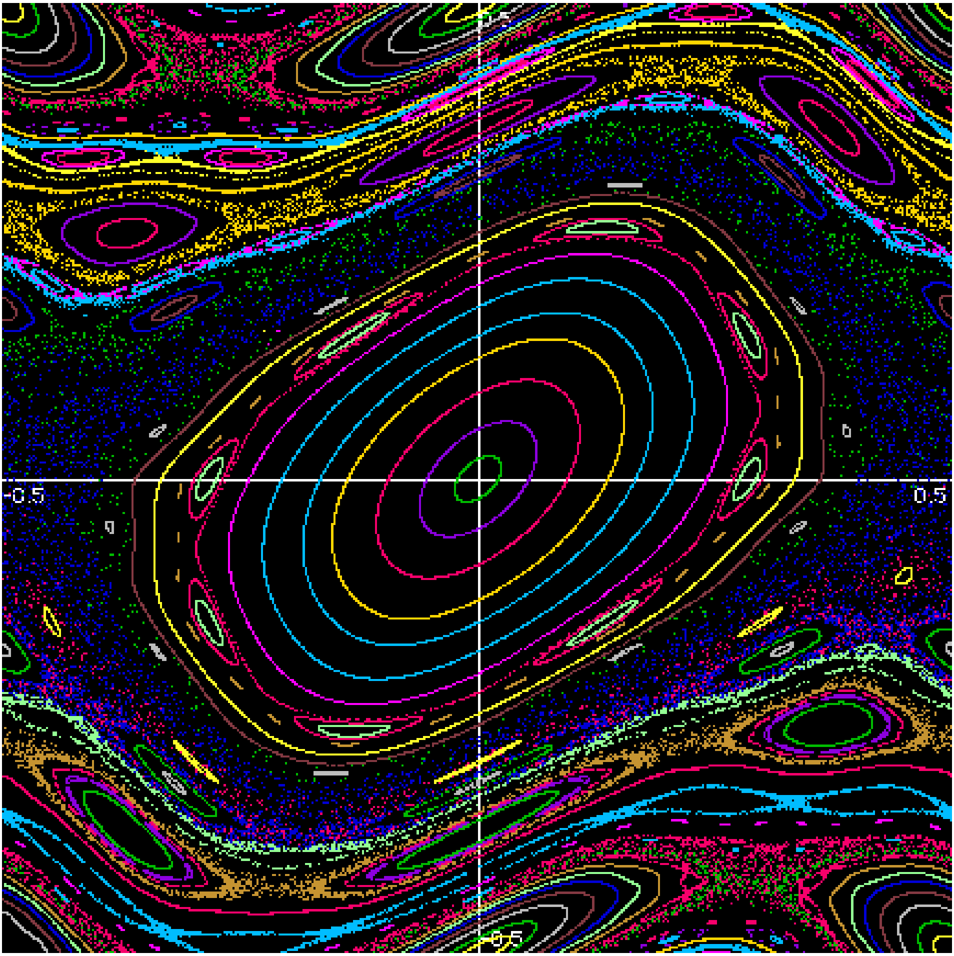
Some orbits with their Lyapunov exponents



dissipative (attracting fixed point) $\lambda < 0$ | dissipative (attracting orbit) $\lambda < 0$ | conservative (neutral fixed point & orbits) $\lambda = 0$

In the diagram below we can see both stable and unstable orbits as exhibited in a discrete dynamical system; the so-called standard map also known as the Cirikov-Taylor map. The closed loops correspond to stable regions with fixed points or fixed periodic points at their centers. The hazy regions are unstable and chaotic.

Sample orbits of the standard map



$$y_{n+1} = y_n - k\sin(2\pi x_n)/2\pi$$
$$x_{n+1} = x_n + y_{n+1}$$
$$k = 0.971635$$
$$-0.5 \le x, y \le 0.5$$

An interesting diversion. Take any arbitrarily small volume in the phase space of a chaotic system. Adjacent points, no matter how close, will diverge to any arbitrary distance and all points will trace out orbits that eventually visit every region of the space. However, the evolved volume will equal the original volume. Despite their peculiar behavior, chaotic systems are conservative. Volume is preserved, but shape is not. Does this also imply that topological properties will remain unchanged? Will the volume send forth connected pseudopodia and evolve like an amoeba, atomize like the liquid ejected from a perfume bottle, or foam up like a piece of Swiss cheese and grow ever more porous? My feeling is that the topology will remain unchanged. The original volume will repeatedly fold in on itself until it acquires a form with infinite crenelated detail. End of diversion.

Given this new measure, let's apply it to the logistic equation and see if it works. The limit form of the equation is a little too abstract for my skill level. Luckily I found an approximation formula in another reference (Nicolis & Prigogine). The Lyapunov exponent can also be found using the formula

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \log_2 \left| \frac{dx_{n+1}}{dx_n} \right|$$

which in the case of the logistic function becomes

$$\lambda \approx \frac{1}{N} \sum_{n=1}^{N} \log_2 |r - 2rx|$$

This number can be calculated using a programmable calculator to a reasonable degree of accuracy by choosing a suitably large "N". I calculated some "lambda" values on a Casio fx-7000G for interesting points on the bifurcation diagram (N = 4000, the first 600 iterates were discarded before counting). The results are listed in the table below and agree with the orbits. You can see there was some disagreement in the sources as to exactly where the chaotic regime begins. Because the calculator used the approximate value of 1 + sqrt(5), the lambda for the superstable 2-cycle is a relatively large negative number, but is not $\lambda = -\infty$ as it should be.

Lyapunov exponents for some values of the logistic equation

r	lambda	comments
1	-1.3552...	start stable fixed point 1 - 1/r
1.99	-5.6489...	
1.999	-8.4734...	
2	$\lambda = -\infty$	superstable fixed point
2.001	-8.4734...	
2.01	-5.6489...	
3	-0.001030...	start stable 2-cycle
3.236067977	-16.5268...	superstable 2-cycle (1 + sqrt(5))
3.449489743	-0.0006876...	start stable 4-cycle (1 + sqrt(6))

3.5699456720	-0.0001471...	start of chaos (Hofstadter)
3.5699457186	+0.0005889...	start of chaos (Dewdney)
3.828427125	-0.0006188...	start stable 3-cycle (1 + sqrt(8))
3.9	+0.6013...	back into chaos
4	+0.8503...	end of interesting region

Speaking of disagreement, the *Scientific American* article that got me started on this whole topic contained the following paragraph:

> I encourage readers to use the algorithm above to calculate the Lyapunov exponent for r equal to 2. Then compare the result with that obtained when r = 3. The first number should be negative, indicating a stable system, and the second number should be positive, a warning of chaos. (Dewdney 179)

Well, I tried those numbers in the equation

$$\lambda = \frac{1}{N}\sum_{n=1}^{N}\log_2|r - 2rx|$$

but I kept getting an error message. Stupid me, I spent several minutes looking for an error in the code not realizing that the mistake was in the instructions. When r = 2 the system quickly settles on to the fixed point of 1/2, which gives r - 2rx = 0. No calculator can find the logarithm of 0 so the program fails. The logistic equation is superstable and thus $\lambda = -\infty$. The second misstatement is that r = 3 is in the chaotic regime. This is most certainly false as this is the location of the first bifurcation. Fact checking is vital when writing for an audience of more than one. Have you found the errors in this paper yet?

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Lyapunov diagrams drawn with Lyapunov

4.4 Lyapunov Space

We have seen how the logistic equation can exhibit behavior reminiscent of a simple harmonic oscillator (equilibrium states, parameter dependent periodicity, Lyapunov stability) and a damped harmonic oscillator (asymptotic stability; over, under, and critical damping). What analog, if any, is there of the driven harmonic oscillator in discrete dynamics? The logistic equation is much richer than the harmonic oscillator in that it can also exhibit chaotic behavior. An exploration of the driven logistic equation should likewise prove interesting.

The differential equation for the damped harmonic oscillator was driven by the addition of a time-dependent force. In a similar manner, a time-dependent population modifier could be tacked on to the left side of the logistic equation. What this would correspond to in our model is uncertain. Individuals would be entering and leaving our population with some kind of regularity. This implies the existence of some source external to the local population. Such behavior is the case with some pack animals. When a pack has too many members some of them (usually the servile males) leave to join other groups. This is one method of driving the logistic equation, but not the one to be discussed in this chapter. I am curious whether anyone has ever explored this possibility.

A second way of looking at the driven and damped components of the harmonic oscillator is as a modification of the system's energy. Driving an oscillator by applying a time-dependent force and damping it by applying a velocity-dependent force does work on the system and changes its energy dynamically. The dissipative term always reduces the system's energy while the driving term can either increase or decrease it. In the logistic equation

$f(x) = rx(1 - x)$ we will mimic this by driving the growth parameter "r" in a time-dependent manner.

Interestingly enough, this method of driving a dynamic system by varying its parameter was first studied in continuous systems. Take the example of a swing. How can one force it to oscillate? The traditional method is by leaning backward or forward in the seat at the extreme points on the arc. This is equivalent to applying a time-dependent force to the swing and is of limited usefulness at large amplitudes. The second method is to stand in the seat and oscillate up and down by squatting. This is equivalent to driving the oscillator by varying its most important parameter; the effective length. This method alone will not get the swing started, but any small deviation from the equilibrium position will be quickly amplified. I once saw this demonstrated by a college professor in a large lecture hall. He was able to increase the amplitude to the point where he was nearly parallel to the ground at the extreme ends of the cycle. This placed him about twelve to fifteen feet above the floor within about a half dozen swings.

Since the growth parameter always seeks to increase the population it differs from the driving force of the oscillator. However, the (1 - x) term acts as a damping factor to restrain the rate of change of the population. It differs from the damping factor "gamma" of the harmonic oscillator in that its power to slow growth is absolute. As long as the growth factor is within the limits of realistic behavior, the population may not exceed the carrying capacity of the environment.

How do we now go about driving the time-dependent parameter of a difference equation; one in which time *per se* is never a factor? Just as with the driven harmonic oscillator, the time-dependent component of the driven logistic equation will mimic the solution of the simple logistic equation. In a discrete dynamical system, periodicity exists as a set of "n" discrete sequenced values that the system will cycle through repeatedly:

$$\{x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n, \ldots\}.$$

The driven logistic equation in difference form will thus be written as

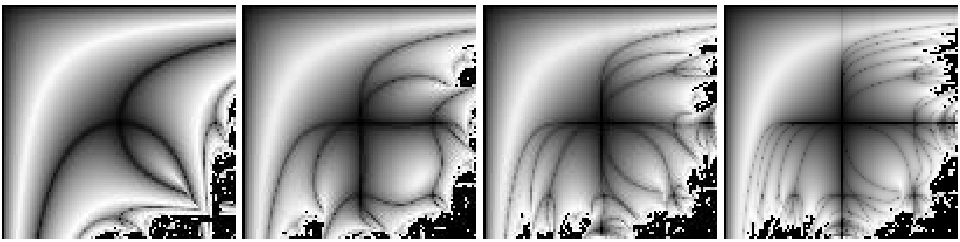
$$A_{n+1} = r_{n \bmod p} A_n (1 - A_n)$$

where $r_0, r_1, \ldots, r_{p-1}$ are the parameters and "p" is the period.

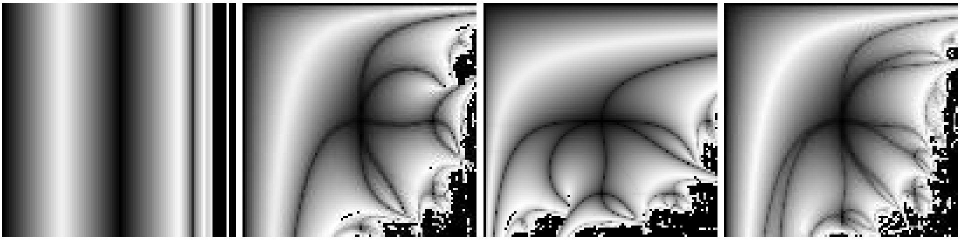
We are now ready to study the behavior of the driven logistic equation. We will do this by examining the Lyapunov exponent of the system as a function of the parameter sequence. For simplicity sake we shall limit ourselves to only two distinct parameter values; call them "a" and "b". Thus {ab}, {aab}, {abaabab} = $\{(aba)^2b\}$ are all acceptable sequences $\{r_n\}$ while {abc} is not. This restriction enables us to plot this information graphically. If we let "a" and "b" be the axis in some parameter space (call it the Lyapunov space) we can then calculate the exponent for all allowable combinations of parameters (those in the interval (0, 4)). The Lyapunov exponent "lambda(a, b)" now behaves as a scalar field like temperature or altitude and can be displayed in much the same manner. To each value we will assign a specific color. The color scheme used will be the one devised by Mario Markus in his 1990 *Computers in Physics* article. Assign white to all points where lambda equals zero and black to all points where lambda is greater than zero. This highlights the transition from order to chaos. For points where lambda is less than zero assign a shade of gray such that values close to zero are nearly white while those close to $\lambda = -\infty$ are nearly black. Assign black again to any point where $\lambda = -\infty$ (*i.e.*, the superstable points and cycles).

A collection of Lyapunov diagrams is presented below. Note that as the number of terms in the sequence increases, so too does the number of superstable arms meeting at a point. Note also that asymmetric sequences produce asymmetric diagrams. Harder to notice, but intuitively obvious is that each diagram has the same cross section through the diagonal line a = b. When this happens, the situation reduces to the bifurcation diagram.

The driven logistic equation in Lyapunov space
Click on an image to see a larger copy



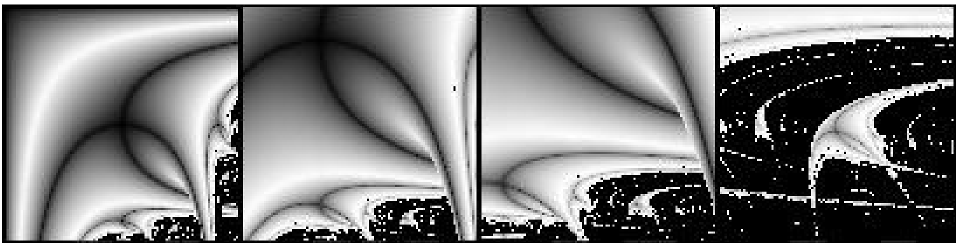
{ab} {aabb} {aaabbb} {aaaaabbbbb}



{a} {aab} {abb} {aabab}

The diagram below is a zoom in Lyapunov space for the sequence {ab}. The first snapshot corresponds to the window with (0, 0) in the upper left hand corner and (4, 4) in the lower right. The black regions below and to the right are chaotic while the largest black stripe is the family of superstable points. The center of the "x" is the point (2, 2). Other stripes correspond to families of superstable n–cycles. The flotsam in the chaotic regime corresponds to stable islands of odd-multiple periodicity. Again, note the self-referential nature of the islands to the whole picture. The blob in the last picture has the same swallow-tail configuration as the entire space as well as its own atoll of stable islets.

A zoom in Lyapunov space for the sequence {ab}
Click the image strip to load the QuickTime movie



The most impressive feature of these diagrams is their three-dimensional appearance. The swallow-tail structure in the last picture, when rendered in more detail, looks like a solid blob and the superstable arms appear to cross over one other. When the sequence is reversed to {ba} the crossing of the arms reverses. At these locations, we have the coexistence of two attractors, each of which may have a different period. Markus found extremely high levels of basin interleaving in more complicated sequences. The ultimate fate of an orbit (that is, to which basin it will attract) depends on which parameter value we start with. There is no analog of this behavior in the driven harmonic oscillator (although this does not preclude such behavior from being found in other continuous systems). We have uncovered a new type of phenomena.

In some cases of overlap, one arm is accompanied by what looks like a shadow. Again, by reversing the sequence, or starting the sequence with the second term instead of the first the direction of overlap will reverse. This is more extreme than just choosing between one basin and another as it includes points that will switch from order (white or gray) to chaos (black). Markus found that by randomly switching between two paired sequences (*e.g.*, between {ab} and {ba} or {abaabb} and {baabba}) orbits in these regions would head for the superstable attractor rather than the chaotic regime. Graphically, the shadows would disappear and the diagram would take on a fuzzy, hybridized appearance. This is the phenomenon of noise-induced order. Chaos with randomness yields stability. Truth is stranger than fiction.

Appendices

A.1 Annotated Bibliography of Printed Resources

There are thousands of printed resources on chaos, fractals, and dimension. These are the sources that inspired me to write this book.

- Barnsley, Michael. *Fractals Everywhere*. San Diego, CA: Academic Press, 1988.
  If you want to really learn about fractals, this is the textbook I recommend. Easy to read.

- Devaney, Robert L. "Overview: Dynamics of Simple Maps" *Chaos and Fractals: The Mathematics Behind the Computer Graphics*. Robert L. Devaney & Linda Keen, editors. Providence, RI: American Mathematical Society, 1988.

- Dewdney, A. K. "Mathematical Recreations: Leaping into Lyapunov Space" *Scientific American*. September 1991: 178–180.

- Gleick, James. *Chaos: Making a New Science*. New York: Viking, 1987.
  More of a history lesson than a mathematics lesson. The companion DOS software package (which I have not tested) is available as shareware from San José State University.

- Harrison, Jenny. "Introduction to Fractals" *Chaos and Fractals: The Mathematics Behind the Computer Graphics*. Robert L. Devaney & Linda Keen, editors. Providence, RI: American Mathematical Society, 1988, 107–126.

- Hocking, John G. & Young, Gail S. *Topology*. New York: Dover, 1961.
  Gail S. Young was my topology professor at Columbia University. The Dover reprint of this textbook is wonderfully inexpensive. I can easily recommend it for its price alone (something around $9 US).

- Hofstadter, Douglas R. *Metamagical Themas: Questing for the Essence of Mind and Pattern*. New York: Basic Books, 1985.
  This was my first introduction to the world of chaos. The description is amazingly simple, but the conclusions are profound. By just goofing around with the parabola, one can generate an entirely new field of mathematics. How amazing is that?

- Keen, Linda "Julia Sets" *Chaos and Fractals: The Mathematics Behind the Computer Graphics*. Robert L. Devaney & Linda Keen, editors. Providence, RI: American Mathematical Society, 1988, 57–74.

- Kline, Morris. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press, 1972.

- Lauwerier, Hans. *Fractals: Endlessly Repeated Geometrical Figures*. translated by Sophia Gill-Hoffstädt. Princeton, NJ: Princeton University Press, 1991.

- Mandelbrot, Benoit B. *The Fractal Geometry of Nature*. revised edition. New York: W. H. Freeman and Company, 1977.
  The book that introduced the Mandelbrot set (called the mu-set). Wanders around a bit, but very entertaining. Hundreds of physical examples are included.

- Markus, Mario. "Chaos in Maps with Continuous and Discontinuous Maxima" *Computers in Physics*. September/October 1990: 481–493.

- Nicolis, Grégoire & Prigogine, Ilya. *Exploring Complexity: An Introduction*. New York: Freeman, 1989.

- Penrose, Roger. *The Emperor's New Mind*. New York: Oxford University

Press, 1989.

- Pickover, Clifford A. *Chaos in Wonderland: Visual Adventures in a Fractal World*. New York: St Martin's, 1995.
  Relevant excepts from this book can be found in the chapter The 15 Most Famous Transcendental Numbers.

- Pickover, Clifford A. "The World of Chaos" *Computers in Physics*. September/October 1990: 460–487.

- Symon, Keith R. *Mechanics*. 3rd edition. Reading, MA: Addison Wesley, 1971.
  My undergraduate mechanics textbook and the resource I used to remind myself of all the mathematics behind the harmonic oscillator.

Appendices

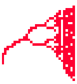
The Chaos Hypertextbook

A.2 Annotated Bibliography of Software Resources

There are thousands of software resources on chaos, fractals, and dimension. I have listed a small sample of the programs available for Mac OS computers. Recommended programs are highlighted in yellow. Some of these were used to create the graphics that were included in this paper. Others were used for inspiration or entertainment. All programs were tested on a Power Mac, 300 MHz 603e running MacOS 8.1. (Note: I have not yet retested these programs with MacOS 8.5.) Appendix 3 also contains links to software archives if you're looking for additional resources.

Chaos (Dynamical Systems)

1-D Chaos Explorer. Matthew Hall. 1992.
Files Associated with 1-D-ChaosExplorer
Bifurcation diagrams, web diagrams, time series, etc. Program your own functions for exploration. Used extensively in Chapter 1.2 and Chapter 1.3. This is a good program for those who want to understand the basic behavior of an iterated system.



Bifurcation. Ronald T. Kneusel. 1995.
Last Known URL bye bye
Faster than 1-D Chaos Explorer, but does half as much. Bifurcation, and time-series only.

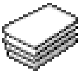
Bouncing Ball. T. Abbott, N. B. Tufillaro, J. P. Reilly. 1987-1993.
Directory of /pub/time-series/Mac
Do you find the quadratic map to be too abstract and the logistic function an oversimplification? All the aspects of the above mentioned functions (steady state, periodic, and chaotic phase space orbits) can be illustrated by a ball bouncing on a vertically oscillating table. It's a great physical application, but I find the program hard to work with and slow. There are too many windows and too many options. It will take some effort to master all of them, so beware.

ChaosPlot. Jason Regier. 1994.
Files Associated with ChaosPlot
Orbit diagrams for continuous systems. Plots the behavior of a damped, driven, anharmonic oscillator. Generates a chaotic path reminiscent of the shadow of a fly on a wall.

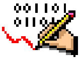
Cliff. John B. Matthews-Gem City Software. 1992.
Files associated with Cliff's World
Cliff is a Macintosh application that iterates Dr. Cliff Pickover's dynamical system and plots the resulting co-ordinate pairs.

Cycle Explorer. James C. Burgess. n.d.
Source Unknown
Click on the bifurcation diagram and draw the corresponding web diagram. Move the parabola around on the web diagram and see the corresponding location on the bifurcation diagram. Very limited with a wise guy interface. Pulling down the options menu gets "No Options" as a reply.

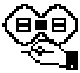
HyperCard Chaos. A.J. Roberts-University of Adelaide. 1991.
Files associated with HyperCard Chaos Stack
The most technical of the HyperCard stacks. A tutorial in dynamical systems with applications and interactive demos. Serious stuff, but doesn't always display well. The interactive demos are a bit erratic also. I loved many of them, but a few were complete mysteries.



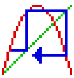
HyperKaos. Fabian Lidman. n.d.
Software by Fabian Lidman
Simple HyperCard stack for iterating one-dimensional functions. Output is a column of numbers. Faster than a programmable calculator, plus you can enter your own functions.



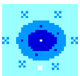
Intelligent Chaos. Fabian Lidman. 1998.
Software by Fabian Lidman
Application for iterating the one-dimensional logistic function. Output is a column of numbers or a time-series graph. Faster than a programmable calculator. I wish I had this when I was doing my research for Chapter 1.1. A version for iterating complex numbers is also available.



Orbit. Dr. Stephen Eubank (Buff Miner, Jim Wiley, Toshi Tajima, Department of Physics, University of Texas, Austin). 1986.
Source Unknown
Note the copyright date. An antique lost in cyberspace. Investigate a variety of well-known 1-dimensional oscillators and 2-dimensional discrete maps. Some of the strange attractors in Chapter 2.1 were drawn on a 68k Mac with this program. Requires a lot of pampering to get it to work on a PPC machine.



Quadratic Map. T. Abbott, N. B. Tufillaro, J. P. Reilly. 1987-1993.
Directory of /pub/time-series/Mac
Draws time-series, bifurcation diagrams (in color), and phase space diagrams (not very useful as far as I'm concerned). Users can't enter their own functions. There are menu items for adding sound, but they don't do anything on my machine.



Std Map. James D. Meiss, University of Colorado. 1994.
Programs of James Meiss
Standard map orbits were drawn using this program. Has other features for exploring two-dimensional iterated maps. A nice piece of code.

Iterated Function Systems (IFS)

Fract. Bob Wiseman-Wiseman Software. 1989-90.
fract-1-0-cpt.hqx

A program for drawing IFS fractals. Comes with several parameter files.

Fractal Attraction. Kevin D. Lee & Yosef Cohen-Sandpiper Software. 1991.
Source Unknown

A reader was kind enough to email me a copy. Draws IFS fractals from a template of polygons. This program was used to create IFS Fractal Movie I which is available on the Web (see below) even though Fractal Attraction is not.

Fractal Lab Kit. Ronald T. Kneusel. 1994
Last Known URL disappeared

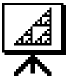
A command-line-driven IFS program for the Mac. Why bother programming for the Mac if you're not going to make use of the Mac interface? I hate this program. The full package also includes one program for Julia set exploration and another for the Mandelbrot set. I can only get them to draw one image and then they crash.

IFS. Paul Bourke. 1989.
Macintosh Software

So far, this is the easiest IFS program to use. I still find that the images generated look nothing like what I predict, however. Uses two different methods: hopalong (which makes images materialize from a haze of dots) and polygon (which is a bit more intuitive).

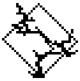
IFS. Stephen Scandalis. 1990.
Files associated with IFS by Scandalis

Another obscure little program. It draws what its name suggests. Nothing special.



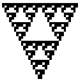
IFS Fractal Movie I. Kevin D. Lee-Sandpiper Software. 1991.
ifs-fractal-movie.hqx

Whatever happened to IFS Fractal Movies II and III? A stand alone program that does nothing other than play a short movie showing an evolving IFS called "The Claw". The midpoint of the movie is a Sierpinski triangle. Created with some long lost program called "Fractal Attraction" (see above). Ah the bad old days: before QuickTime, before MPEG, before the Web.



Kaos. Reinoud Lamberts. 1990.
kaos-004r.hqx

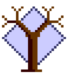
Kaos produces IFS images that are quite unique. Painfully slow! After running half an hour on a 300 MHz 603e PPC, I had a smudge that occupied one-third of the screen. An interesting looking smudge, but it wasn't worth the wait. Kaos requires an FPU. The author has a home page, but it doesn't mention the program.



Sierpinski-Triangeln. Martin Wiss. 1997.
sierpinski-triangeln-1-swe.hqx

A minimal little program that draws the Sierpinski Triangle. You can play around with the parameters, but that's it. Written in Norwegian, but simple enough for Americans to figure out.

Lindenmayer Systems



Fractal Trees. Simon Woodside. 1997.
Download, Abstract

A really bad program. Draws fractal trees using an IFS, but has no interface. None! Its parameters can only be modified in the source code. My advice, don't waste your time.

L-Systems. Paul Bourke. 1991.
Macintosh Software
Draws recursively defined fractals like the type shown in Chapter 3.2 (Peano monster, Koch coastline, Sierpinski gasket, etc.). The L-System Manual is worth reading for its own sake. Also available in a 3D version that crashes every time I launch it.

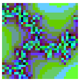
LSystems. Bryan Horling. 1996.
Senior Research: Lindenmayer-Systems
Renders fractals from simple recursion instructions. Colorful and visually appealing. The author has also written a nice, compact paper on branching systems in nature (plants, corals, etc.).

PFG (Plant and Fractal Generator). Przemyslaw Prusinkiewicz. 1988-92.
Directory of /pub/projects/pfg
Draws fractal images and plant-like branching structures using L-systems with "turtle interpretation". Fast, but difficult to work with. No options. No interface for producing your own parameter sets (must be coded by hand using a text editor).

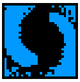
Mandelbrot & Julia Sets

ab Fractal. Eden Software (L. Pieniazek). 1994.
ab-fractal-1.2.sit.hqx
I don't have time for this. It draws the Mandelbrot set, Julia sets, and maybe some other sets, but unbelievably slowly. I was insulted that anyone would ask a shareware fee for anything so bad as ab Fractal. Requires an FPU.

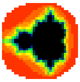
Aros Fractals. Aros Magic Research. 1996.
Aros Fractals
Mandelbrot set, Newton's method, and a third fractal type I've never seen before that looks something like a moire pattern. Zoom in, but not out. Animated colors. Unusual interface with limited options. Also available in a Windows version.



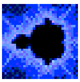
CFG (ColorFractalGenerator). John Schlack. 1990-95.
Shareware: Color Fractal Generator
Mandelbrot sets, Julia sets, random walk, etc. A basic rendering program for drawing color fractals. Unregistered versions aren't PPC-native and run slowly.



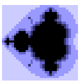
EasyFractPPC. Alessandro Guzzini. 1997.
easyfract-ppc-1.1.sit.hqx
Easy, yes, but there are better programs out there. Zoom into the Mandelbrot set, play with the colors, but that's all.



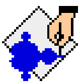
EscapePPC. Graham Anderson. 1998.
Escape Webpage
Just a basic program for drawing fractals: 25 different types, including the standard Mandelbrot and Julia sets. Not particularly fast or interesting, but not a dinosaur either (note the relatively recent copyright date). Batch processing feature for creating movies. Not to be confused with EscapeFractals (see below).



Floating Fractals. Adam Smith. 1994.
Files Associated with Floating Fractals
Zoom in and explore 11 different fractals (half of them are variations on the Mandelbrot set). Fast and easy to use.



Fractal Artist. Alexei Lebedev-Phronesis Software. 1992.
Last Known URL Long gone.
Another basic Mandelbrot-Julia set explorer, now lost in cyberspace. It was an obscure program in the first place.



Fractal Domains (formerly FracPPC). Dennis C. De Mars. 1994-97.
Fractal Domains Home Page
Explore the Mandelbrot set. Switch to the accompanying Julia set. Rather plain interface with lots of control panels floating around. Not my favorite, but it appears to be quite popular.

Fractal Observatory. Marcio Luis Teixeira-Trilobyte Software. 1990.
MarcioT's Software Archive
Another program that doesn't use the Mac interface. Halftone grays meant to be viewed on a black and white monitor. Looks and behaves like SuperMandelZoom.

Fractal Studio (formerly Fractals). Keiron Liddle -Aftex Software. 1998.
Fractal Studio Home Page
Draws 30 different fractals. Has a ton of different options to play with. I haven't had the energy to fully explore this monster.

Fractastic! Jake Olevsky. 1998
Jake Olevsky: Fractals
The best piece of eye candy to hit the Mac in a long time. Draws eighteen different fractals including Henon Attractors, Julia sets, Newton's method, and Mandelbrot sets up to the sixth power and does it fast. Includes dynamic zooming and numerous coloring methods with crazy-fast animation as an option. Requires the Draw Sprockets library, available free from Apple.

f_zoom. Andreas Warnke. 1996.
LEO: AFA record for f_zoom.sit
"Zoom in die Mandelbrotmenge (je nach Rechenleistung sogar in Echtzeit)." A cute little nothing of a program for dynamical zooming into the Mandelbrot set. A rare piece of überminimalism from Germany.

Intelligent Mandelbrot. Fabian Lidman. 1998.
Software by Fabian Lidman
A unique application for iterating the complex quadratic map. Dig around in the dirt of the Mandelbrot set. Explore the actual numbers involved in generating the pictures produced by other programs. Produces two columns of numbers as output. Faster than a programmable calculator.

JLB's Dirty Mandelbrot. Jean-Luc Brousseau. 1998.
mandelbrots-fractal.h+

Somebody's weekend project. A decent job for two hours of work, but there's nothing special about this program.

JavaQuat. Garr Lystad. 1997.
Lystad's Fractal Top, Home of JavaQuat

A Java applet for exploring Mandelbrot sets, Julia sets, and the sets in between. Uses quaternions: complex-complex numbers of four real parameters! Stretch your visualization muscles as you view two-dimensional slices through the four-dimensional mother of all sets. Not as smooth running as a well-written, stand-alone application would be. Oversized interface makes it hard to work with on smaller monitors.

Julia! Julie Mitchell. 1993.
Files associated with Julia!

Another Julia-Mandelbrot explorer. Not fast or sophisticated. In fact, it's downright slow. Interlaced graphics and minimal options. There are better programs available.

Julia 'O Matic. Jim Burgess. n.d.
juliaomatic1.0.sit.hqx

Tiny windows. No documentation. No explanation. Sort of fast, but who cares? Another minimal Julia-Mandelbrot explorer.

Julia's Dream. Reinoud Lamberts. 1991.
Files associated with Julia's Dream

Many of the Julia sets in this paper were drawn using this program. Generates real-time images of Julia sets as you roam around the complex plane with your cursor. The author has a home page, but it doesn't mention the program. (Julia's Dream was also the name of a pizza: basil pesto, ricotta and mozzarella, topped with garlic, broccoli, and spinach. $9.99 for a 12" pie. 804-978-7898.)

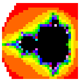
Julia's Nightmare. Ben Davenport. 1995.
Files associated with Julia's Nightmare
A full-color sequel to Julia's Dream. Nothing else is known about this program.

MandelAcid. Derek Greenberg-The Bone Factory, 1993.
Files associated with MandelAcid
"A video drug for the Macintosh." Eye candy, pure and simple. Way cool, but useless for those interested in mathematics. No theory behind what you're looking at. (Peculiar sideline. Read "Virtual VIKKI's Twilight Gallery". MandelAcid used to clothe a computer-generated animal in tie dye thus rendering it safe for viewing by children. You can't get much weirder than this.)

MandelBrain. Danny Brewer. 1994.
MandelBrain
Explore the Mandelbrot set. Comes with more preset color palettes than any other program.

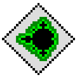
Mandelbrot Fractal Companion. Evan T. Yeager. 1993.
mandelfractalcomp.cpt.hqx
A brief HyperCard tutorial of complex numbers, fractals, the Mandelbrot set, and how to construct a computer program. In living color (something quite rare in HyperCard).

MandelBrowser, Mandella, MicroMandella. Jesse Jones. 1990-95.
Files associated with MandelBrowser
Mandelbrot sets, Julia sets, Newton's method, and a few others. Over 50 different sets and numerous color schemes. Fast, PPC-native code. Mandella is an older 68k version that also draws strange attractors (the Ikeda attractor in Chapter 2.1, for example). MicroMandella is a stripped-down version of the 68k program.



MandelMovie. Michael Larsen-Dynamic Software. 1990-95.
Dynamic Software Products: Software

Mandelbrot-Julia hybrid and Julia cascade movies in section 2.3 were rendered in 1992 at the University of Wisconsin-Milwaukee, School of Education using this program. Haven't used it since, so I can't comment on it. Dynamic Software offers other chaos and fractal programs.

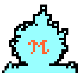


MandelScope, MandelMaker. Anthony S. Ku. 1993.
mandelscope.maker1.0.sea.hqx

Explore the Mandelbrot set with MandelScope and then draw it using MandelMaker. Why not combine these features into one program? MandelScope has an unusual interface that might have seemed clever at one time, but it looks ridiculous now compared to the hot rod programs out there now. MandelMaker uses a divide and conquer algorithm that should speed things up, but it still runs slowly. Doesn't use the symmetry of the Mandelbrot set across the real axis to save time either. Both programs require an FPU.

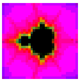
MandelTV. Ed Ludwig & Ken Abbott-Abbott Systems, Inc. 1990.
mandeltv1.0.cpt.hqx

A relic from the days of System 6. MandelTV is a *desk accessory* for exploring the Mandelbrot set. Requires an FPU.

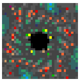


MandelZot. David Platt-Think Technologies. 1988-1993.
mandelzot4.0.sea.hqx

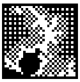
Most of the Mandelbrot sets and some of the Julia sets in this paper were drawn using this program. I have been using it since 1991 so I'm used to it. (My God! How old am I?) Apparently, a commercial version called FractalMagic is also available from Sintar Software, but I can't confirm this. Comes with extensions for drawing Markus-Lyapunov fractals that I can't get to work.



Object Mandelbrot. Bryan Prusha. 1996.
Object Mandelbrot 1.0
A personal favorite. Zoom into the Mandelbrot set, draw the corresponding Julia set. Fantastically fast, with dynamic zooming. Hold down the mouse and zoom away. Keep zooming until you reach the limits of your computer's numeric resolution.



PowerXplorer. Allesandro Levi Montalcini. 1996.
ALM Software
Bare bones simple. No interface, no documentation, no options. Draw a box around the region you want to explore and let go. PowerExplorer zooms you in. Intended primarily for testing computing speed. Has a bug that replaces the menu bar with a blank region in full screen mode.



Super MandelZoom. Robert P. Manufo. 1988.
super.mandelzoom1.06.sea.hqx
Another antique still floating around. Meant to be viewed on a Mac SE or thereabouts. Uses halftones instead of grays.

XaoS Fractal Viewer. Jan Hubicka. 1996.
XaoS: Real-Time Fractal Zoomer
Mandelbrot sets of powers 2-6. Zoom in dynamically, then switch to the Julia set at the same level of magnification and the same point. Nice for showing the quasi-self-similarity between the two sets. Other fractals included: "octal", Newton's method, "Barnsley", and "Phoenix". Comes with a really neat option that remaps the complex plane. Very fast, optimized code that doesn't try to redraw the entire window. Intentionally ugly interface takes some getting used to, however.

Music

BifurcationOscillator, **Logistic Synth**. Jae Ho Chang. 1996.
bifurcation-oscillator-10b2.hqx, logistic-synth-02-ppc.hqx

Orbit diagram and chaotic music generator. Watch the evolution of an orbit and then transform it into a music-like series of notes saved in AIFF format. Logistic Synth is an equivalent program for generating sounds in real time. Although these programs have since been shelved, the author has other new and interesting projects on-line at his home page.

BirdSong Engine. David Benz. 1993.
fractal-birdsongs-1.2.sit.hqx

Clever HyperCard stack that converts L-System-type fractals into bird songs.

Chaos Theory. Mike Atanasio-Wild Card Software. n.d.
chaos-theory.sit.hqx

HyperCard stack. Listen to the time-series of the logistic equation. A one card stack. No date, but it appears that it was created in 1990.

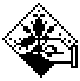
ChaoticPianola. The Boltzmann Toy Factory (Lars Rosenberg). 1998.
Sounds from the Realm of Chaos

Listen to the behavior of the iterated logistic function played on the QuickTime Roland digital piano. A different approach to the exploration of a dynamical system. Sounds like Philip Glass when periodic and Schoenberg when chaotic. Since there is a sort of structure to the chaotic regime, ChaoticPianola is briefly listenable from time to time. Also plays a random function for comparison (and to reseed a stuck or stale pattern).

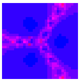
MandelMusic. Roger R. Espinosa, Donna Iadipaolo, Jim Brunberg. n.d.
mandelmusic.cpt.hqx

HyperCard stack. They call it a chaotic music generator, but I can't confirm this. No mathematical description of what it actually does. Your choice of 12 different instruments.

Newton's Method



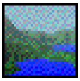
NewtonFractal. Stefan Messmer. 1991.
Newton-Fractals (Theory) or Newton-Fraktale (Theorie)
Painfully slow. Stay away from this program if you value your time. I didn't fully evaluate this problem as I got tired of waiting for it.



Object Newton. Bryan Prusha. 1996.
Last Known URL A nice program that vanished
Object Newton allows you to explore Julia sets created using Newton's root-approximation method. Fast interface like Object Mandelbrot.

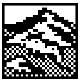
Terrain & Landscape Modeling

FracHill, FracBlob. Paul Bourke. 1991.
Macintosh Software
Wire frame fractal terrains. Slow, requires an FPU, and doesn't look very good. I'm not impressed. FracHill models on a plane while FracBlob models on to a sphere, but the author seems to have forgotten about the program.



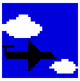
Fractal! Ed Rotberg-Gonzo Systems. 1992-93.
fractal-1.4-ppc.sit.hqx
Draws interesting snowcapped mountains with blue lakes in the foreground. Play with lighting and color schemes. Use it to render a desktop background if you wish.

Fractal Contours. Jim Cathey. 1985.
fractalcontours1.0.sit.hqx
Crash Warning. Gives "Error 2" when launched, even with PowerFPU. Using ResEdit, I found out it's a fractal terrain modeler of some sort based on algorithms developed by LucasFilm, Ltd. Looking at the icon, I'd say someone is playing a joke on us. Tell me what you think it looks like.



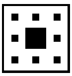
Fractal Islands. Scott Berfield-Parity Productions. November 1985.
fractisle.hqx
The oldest piece of shareware on the Net. Guess what? It crashes after drawing one wireframe pseudo-fractal surface.

Matt's Fract. Matson Dawson. 1995.
Files associated with Matt's Fract
Fly around over a fractal-generated terrain. Not detailed or realistic in any way. Plenty fun, though.



Venus, New Venus. Oleg. 1997.
Venus
Someone spent a lot of time working on this, I'm sure, but I find it amateurish and stupid. The ReadMe describes it as "Virtual circling around through the clouds." Looks like a bunch of nothing to me. Ironically, New Venus is an older version with controls.

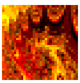
Miscellaneous



Carpet. Kevin Whitley-Think Technologies. 1987-88.
mycarpet2-0-sit.hqx
A simple program for stamping carpet or gasket fractals. Ancient, but it still runs.

?

Cubic Oscillator Explorer. Bruno Degazio. 1995.
Cubic Oscillator Explorer
Can't download it for review. "Communications exception (-244)." Would someone out there send me a copy?



EscapeFractals. Yves Meynard. 1996.
Download, Abstract
This program draws fractals using an escape-time method developed by Clifford Pickover. Visually appealing and fast. Not to be confused with EscapePPC (see above).

Fractal Dimension Calculator. Paul Bourke. 1993.
FDC
Hausdorff-Besicovitch dimension calculations using the box-counting method. Can be slow. Data points in Chapter 3.3 were calculated with this program. The accompanying instruction manual is worth reading for its own sake. Also includes a 3D version that I haven't tested.

Fractal Explorer. Peter Stone. 1999.
Fractal Explorer Home Page
Easily the largest collection of preset fractals: Mandelbrot sets, Julia sets, escape fractals, Newton's method, quaternions, and others. Enter your own complex functions and see what happens. Fairly fast, but the interface is too clumsy for this to be a fun program. Zooming in is especially tiresome.

Fractal Wizard. Thomas Okken. 1992
fractalwizard1.6.cpt.hqx
Mandelbrot sets, Julia sets, Newton's method, IFS, and a few others. Quite a range of different sets, but it's basically another slow antique. Has a nice pull down menus briefly explaining the mathematics. Requires an FPU.

Fraxious. Alan Smith. 1995.
Files associated with Fraxious
Mandelbrot and Julia sets, Newton's method, Henon and other Strange attractors, Brownian motion, and Lyapunov fractals. Crashes whenever I try to draw a strange attractor. The ReadMe admits it's a buggy program. Not PPC optimized, so it's slow.

Halley Maps. Yves Meynard. 1997.
Download, Abstract
This program produces interesting fractals using Halley's root-approximation method for equations up to the twelfth degree. Windows must all be square and less that 512 pixels in size. No animated colors or dynamic zooming and yet I still like it.

Iterative Functions. Harold Brochmann. 1992.
Source Unknown
An odd suite of six programs: diffusion limited aggregation, real orbits, complex orbits, the Mandelbrot set, logistic equation, and population dynamics. Lousy interface that makes no use of Mac features. A time capsule from the bad old days of programming. Great in terms of content, however. Someone should rewrite this suite and wrap it in an up-to-date interface.

Lyapunov. Andrea Pellizzon. 1994
Files associated with Lyapunov
Lyapunov diagrams in this paper were drawn using this program as was the Lyapunov zoom movie in section 4.3 (included in the package). This is the only program I know dedicated to drawing this kind of fractal.

Persian Rug. Paul Cantrell. 1996.
Last known URL Removed from the Info-Mac archive.
Persian Rug is a control panel that generates random fractal patterns for your desktop. Don't like your current desktop? Open the control panel and click the "change" button. A really great idea. Unfortunately, there's a bug in it that causes my machine to hang.

After Dark Modules

Clouds. Robert Geisler. 1993-95.
clouds2.01.sit.hqx
This module draws fractal clouds that drift across your desktop. Nicely done.

Fractal Forest. Scott Armitage-Berkeley Systems Inc. 1992.
Last Known URL Where's it gone? You tell me.
Draws simple, cartoon-like fractal trees on your desktop.

Fractals. Alessandro Levi Montalcini. 1992.
fractals1.33.cpt.hqx

This module draws regions of the Mandelbrot set which are then saved as pict files. An interesting idea, but the old source code makes this module run slowly on PPC Macs. Part of Fractals Bundle 1.33: a suite of programs designed to run on 68k Macs with a math coprocessor. Comes with a program for viewing the picts and a mover based on the old Font/DA mover. Both of these applications are of little value in the age of OS 8.

IFS Dream. Justin Sands. 1994.
ifsdream1.0.cpt.hqx

Draws animated IFS fractals from evolving parameter sets.

Newton Map, Newton-EXP. Huangxin Wang. 1994.
newtonmap.sit.hqx

Far too slow. Each requires an FPU. I assume they draw Julia sets using Newton's method, but I couldn't stand the wait.

Ocean Child. Paul Cantrell. 1994.
oceanchild1.01.cpt.hqx

Perpetual, random Julia cascade with options for different surreal effects. An excellent piece of eye candy.

Planetmaker. Adam Miller. 1992.
Last Known URL It's history.

Draws green on blue patterns that look something like continents and oceans. It's a real stretch to call it a "fractal planet maker," however. Very slow.

Sierpinski Gasket. WinterBright Software (David Thompson). 1992-93.
sierpinskisgasket1.1.sit.hqx

Crash Warning. Gives "Error 10" when launched, even with PowerFPU.

Terrain Maker. Jakub Buchowski & Adam Miller. 1991.
terrainmaker2.0.sit.hqx
Fractal terrain modeler. Old-fashioned looking color scheme and resolution.

Appendices

The Chaos Hypertextbook

prev | index | next

chaos, dimension, faq, feigenbaum, fractals, image-compression, ifs, landscapes, l-systems, logistic-equation, mandelbrot-julia, metric-spaces music, newsgroups, newton, periodicals, personalities, science, software, strange-attractors, miscellaneous

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A.3 Topical Index of Internet Resources

Chaos (Dynamical Systems)

- Applied Chaos Lab, Georgia Tech
- Caos - Chaos, A. Carden, Universidad de Los Andes
- Center for Nonlinear Dynamics, University of Texas, Austin
- Center for Nonlinear Studies, Los Alamos National Laboratory
- Chaos, Complex Systems Virtual Library, Charles Sturt University
- Chaos Demonstrations, Michael Cross, Caltech
- Chaos Group, University of Maryland, College Park
- Chaos Homepage, Andrew Ho, University of Illinois Urbana-Champagne
- Chaos MetaLink, Industrial Street Productions
- Chaos Research Group, University of Tennessee, Knoxville
- Chaos: The Limits of Predictability, Heureka: The Finnish Science Center
- Chaos Theory, Manus J. Donahue III
- Chaos Theory, Study Web
- Chaotic Dynamics of f(x)=cxe^x, Loren Hoffman

- Chaotic Systems Textbook for PFP 96, University of Pennsylvania
- Dynamical Systems and Technology Project, Robert L. Devaney, Boston University
- Kelleen Farrell, a suite of web sites devoted to chaos and nonlinear dynamics
- Mainz Nonlinear Dynamics Bibliography
- Meddling in the Affairs of Infinity, Shohdy Nagib, (video feedback)
- Messy Futures and Global Brains, Gottfried Mayer-Kress, Santa Fe Institute
- Nonlinear Dynamics, Complex Systems Virtual Library, Charles Sturt University
- Non-Linear Lab, Blair Fraser, University of Western Ontario
- Nonlinear Systems Laboratory, Massachusetts Institute of Technology
- Random Acts of Finance: Chaos Theory, Budget Practice, John Allen Paulos
- Theory and Practice of Chaos, Dale Winter, University of Michigan

Dimension

- Benoit Fractal Analysis Software, Trusoft International
- Computing the Hausdorff Measure and Dimension, Mark Krosky, Cornell University
- Exploring Fractal Dimensions of Strictly Self-Similar Fractals, Mary Ann Connors, University of Massachusetts, Amherst
- Fractal Dimension Calculator User's Guide, Paul Bourke, Mental Health Research Institute of Victoria
- FracTop V0.2, Charles Sturt University (calculates fractal dimension of gif files)

FAQs (some of the many copies)

- sci.fractals
  - Japan
  - Netherlands
  - USA
  - http://www.faqs.org/faqs/sci/fractals-faq/
- sci.nonlinear
  - Australia
  - UK
  - USA
  - http://www.faqs.org/faqs/sci/nonlinear-faq/

Feigenbaum Constants (Bifurcation Diagrams)

- 15 Most Famous Transcendental Numbers, Cliff Pickover, University of Wisconsin
- Feigenbaum Constant, A. S. Buch, University of Aarhus
- Feigenbaum Constant, Steven Finch, MathSoft

Fractals

- About Fract-ED, Douglas Martin, EALSoft
- chaotic n-space network, Jon Camp & Ben Martin
- Chopping Broccoli, Evan Glazer, Northfield Township High School
- Fantastic Fractals, The Why Files, University of Wisconsin, Madison
- Fractal Pages at Thinks.com
- Fractal Links Page, Chaffey High School, Ontario, California
- Fractal Microscope, Shodor Foundation
- Fractal World, Erick Hanson, Lock Haven High School
- Fractals, Paul Bourke, Mental Health Research Institute of Victoria

- Fractals, Brad Johnson
- Fractals, J.P. Louvet, Université Bordeaux
- Fractals, Complex Systems Virtual Library, Charles Sturt University
- Fractals, Mathematics Archives, University of Tennessee, Knoxville
- Fractals, MathsNet, Anglia Multimedia
- Fractals Lesson, Cynthia Lanius, Rice University
- Fractals and Scale, David G. Green, Charles Sturt University
- Frequency for Sci.Fractals, PHOAKS (People Helping One Another Know Stuff)
- Geometry Junkyard: Fractals, David Eppstein, University of California, Irvine
- Groupe Fractales, Institut national de recherche en informatique et en automatique (INRIA)
- Human Consciousness: Its Fractal Nature, John Allen Paulos
- McTaylor's Fractal Archives, Mount Allison University
- Paper-Folding Fractals, Joel Castellanos, Rice University
- Spanky Fractal Database, Noel Griffin, TRIUMF
- Twin Dragon Applet, Jelena Kovacevic, Bell Labs (Lucent Technologies)
- Zona Fractal

Image Compression

- Altamira Group
- Fractal Image Coding, Video Coding Group, Bath
- Fractal Image Compression Bibliography, Brendt Wohlberg, University of Cape Town
- Fractal Image Encoding, Yuval Fisher, University for California, San Diego
- Genetic Algorithms for Fractal Image and Image Sequence Compression, Isaac Rudomin, Instituto Tecnológico de Estudios Superiores de Monterrey

- IFS Fractal Image Compression, Valerio Verrando & Giovambattista Pulcini
- Iterated Function Systems for Visualization, Craig M. Wittenbrink, University of California, Santa Cruz
- Iterated Systems, Inc.
- Waterloo Fractal Compression Project, Edward R. Vrscay, University of Waterloo

Iterated Function Systems (IFS)

- Fractals: Not by the Numbers, Gary Kerbaugh, East Carolina University
- IFS Playground, Otmar Lendl, Universität Salzburg
- Interactive Fractal Generation Using Iterative Function Systems, Richard L. Bowman, Bridgewater College
- Just Another Fractal Generator, Brighten Godfrey
- Nonlinear Iterated Function Systems, Eduard Gröller, Vienna University of Technology

Lindenmayer Systems (L-Systems)

- Java Fractal Tree, David Hanna
- L-Systems, Complex Systems Virtual Library, Charles Sturt University
- L-Systems Tutorial, David G. Green, Charles Sturt University
- L-Systems Group at Trinity, Trinity College
- L-Systems Software, Biological Modeling and Visualization, University of Calgary
- Laurens Lapre's Lparser Links

Logistic Equation

- Emergence of Chaos, Alexander Bogomolny
- Generalized Logistic Equation, Douglas J. Ingalls

Mandelbrot & Julia Sets

- Alberto Strumia, Università di Bari no reply
- Area of the Mandelbrot set, David Eppstein, University of California, Irvine
- Fractal Explorer Kit, Tom Harrington
- Fractal Explorer: Mandelbrot and Julia sets, Fabio Cesari
- Fun With Fractals, Wildfire Communications
- Guide to the Mandelbrot Set, Paul Derbyshire, Chaos-Forschungsgruppe, Universität Zürich
- Java Julia Set Generator, Mark McClure, Univerisity of North Carolina at Asheville
- Keith Lynn, University of South Alabama
- Mand.java, Bob Jamison, Java Boutique
- Mandel, Ken Shirriff, Sun Microsystems
- Mandelbrot Exhibition, Virtual Museum of Computing, Oxford University
- Mandelbrot Explorer, Panagiotis Christias, National Technical University of Athens
- Mandelbrot page of Zs.Zsoldos, Zsolt Zsoldos
- Mandelbrot Set, Franceway
- Mandelbrot Set, Simon Arthur
- Mu-Ency: The Encyclopedia of the Mandelbrot Set, Robert Munafo
- My own fractal-page, Frode Gill, Høgskolen i Agder
- Vagfractals (MandelMaster), Martin Sandin

Metric Spaces

- Taxicab Geometry Project, Norma Shunta, Lorrie Graff, Julie VanBelkum; Grand Valley State University

Music: Chaotic & Fractal

- Experiments with fractal music
- Fractal Music Project

Newsgroups

- alt.binaries.pictures.fractals
- alt.chaos not much about mathematics here
- alt.fractals
- alt.fractals.pictures
- bit.listserv.frac-l no such group
- comp.theory.dynamic-sys
- geometry.software.dynamic
- sci.chaos no such group
- sci.fractals
- sci.nonlinear
- sfnet.tiede.nonlinear very little traffic
- uw.mail-list.fractals very little traffic
- z-netz.alt.fractint very little traffic

Newton's Method

- Méthode de Newton, Alain Goudey
- Newton's Method, H. Edward Donley, Indiana University of Pennsylvania
- Newton's Method and Self-Similarity, Douglas J. Ingalls

Periodicals

- Complexity International no reply
- Nonlinear Science Today/Journal of Nonlinear Science

Personalities

- Robert L. Devaney, Boston University
- Mitchell Feigenbaum, Rockefeller University
  - Center for Studies in Physics and Biology
  - Mitchell Feigenbaum Lab
- James Gleick
  - Chaos: The Software
- Clifford A. Pickover, University of Wisconsin, Madison
- Clint Sprott, University of Wisconsin, Madison

Science

- Abteilung für Nichtlineare Physik, Technische Hochschule, Darmstadt
- Celestial Mechanics and the Stability Problem, Jeff Suzuki, Boston University
- Centre for Chaos And Turbulence Studies (CATS), University of Copenhagen
- Chaos and Mixing Group, Northwestern University
- Chaotic Mixing due to Separatrix Crossing, John R. Cary, University of Colorado
- Classicals and Quantum Chaos (ChaosBook), Predrag Cvitanovic', Niels Bohr Institute
- Combustion Chaos Group, University of Texas at San Antonio
- Dance of Chance (Polymer Science), Boston University
- Fractal Clouds Info, Robert F. Cahalan, NASA-Goddard
- Fractal Patterns of Seaweed Settlement, L.M. Emmerson & A.J. Roberts, University of Southern Queensland
- Gibb's Phenomena and Self-similarity, Douglas J. Ingalls
- Nonlinear Dynamics, The Net Advance of Physics, Massachusetts Institute of Technology

- Nonlinear Dynamics in Ferromagnetic Resonance, Ohio State University
- Society for Chaos Theory in Psychology and Life Sciences, Vanderbilt University
- Solar System is in Chaos, Nigel Bunce and Jim Hunt, University of Guelph
- Xmorphia (Morphogenesis from a Reaction-Diffusion System), Roy Williams, California Institute of Technology

Software Archives (Java)

- Gamelan: Java: Education: Math: Fractals

Software Archives (Macintosh)

- Link Everything Online (LEO)
  - pub/comp/os/macintosh/leo/science/fractal/
- Mathematics Archives, University of Tennessee, Knoxville
  - software/mac/fractals/
- Spanky Fractal Database, Noel Griffin, TRIUMF
  - pub/fractals/programs/mac/
- Washington University, St. Louis
  - edu/math/software/mac/fractals

Strange Attractors

- 3D Strange Attractors and Similar Objects, Tim Stilson, Stanford University
- Strange Attractor Search, Clint Sprott, University of Wisconsin
- Lorenz
  - Edward N. Lorentz, Exploratorium
  - Lorenz Attractor, Andrew Ho, University of Illinois Urbana-Champaign
  - Lorenz Butterfly, Exploratorium

- Henon
  - Henon Attractor, Andrew Ho, University of Illinois Urbana-Champaign

Terrain & Landscape Modelling

- Java-mountains page, Stephen P. Booth, Edinburgh Parallel Computing Centre (EPCC)

Miscellaneous

- Art Matrix
- AVirtualSpaceTimeTravelMachine, Jean-Francois Colonna, École Polytechnique
- Chaos and Fractals, Joakim Linde, Chalmers Tekniska Högskola
- Fractals Calendar, Simon Fraser University
- Fractals & Chaos, Meta, inc.
- Gallery of Mathematics, Loughborough University
- Harold Brochmann
- Java Gallery of Interactive On-Line Geometry, University of Minnesota
- Java Fractals, James Henstridge
- Lexicon, Jim Crutchfield & Tom Humphrey, Exploratorium
- Review of Literature on Chaos, Fractals, and Non-Linear Dynamics, Steve Lee
- Scott's Nonlinear Science Hotlist, Scott Peckham, University of Colorado
- Stock market timing, Is it real or is it bull shit?
- ThinkQuest
  - Chaos Experience
  - Chaos & Fractals
  - Chaos Theory, Dynamic Systems, and Fractal Geometry
  - Fantastic Fractals

    - The Fractory: An Interactive Tool for Creating and Exploring Fractals
    - The FUNKtion
    - InterFACE (Internet Fractal and Chaos Education)
    - Making Order Out of Chaos
    - Yarra Valley Fractals
- Yahoo!
    - Chaos
    - Fractals