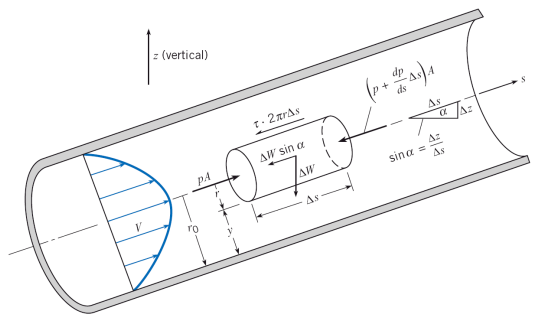
Chapter 10

FLOW IN CONDUITS

Fluid Mechanics, Spring Term 2009

Shear stress distribution across a pipe section



For steady, uniform flow, the momentum balance in s for the fluid cylinder yields

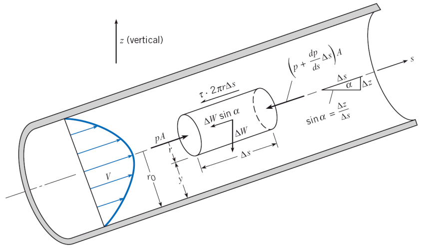
$$\sum F_s = F_{\text{pressure}} + F_{\text{gravity}} + F_{\text{viscous}} = 0$$

$$\Rightarrow \quad pA - \left(p + \frac{dp}{ds}\Delta s\right) A - \Delta W \sin\alpha - \tau\,(2\pi r)\,\Delta s = 0$$

with $\Delta W = \gamma A \Delta s$

and $\sin\alpha = \dfrac{dz}{ds}$

we solve for τ to get:



$$\boxed{\tau = \frac{r}{2}\left[-\frac{d}{ds}(p + \gamma z)\right]}$$

regardless of whether flow is laminar or turbulent.

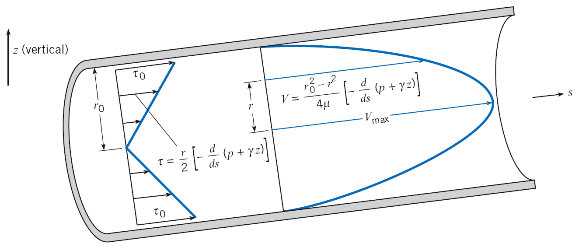
(Technically, turbulent flow is neither uniform nor steady, and there are accelerations; we neglect this).

Velocity for laminar flow in pipes

Using the result for τ, we substitute

$$\tau = \mu \frac{dV}{dy}$$

$$= -\mu \frac{dV}{dr}$$



$$\Rightarrow \quad -\mu \frac{dV}{dr} = \frac{r}{2}\left[-\frac{d}{ds}(p + \gamma z)\right]$$

Integration yields $\quad V = -\dfrac{r^2}{4\mu}\left[-\dfrac{d}{ds}(p + \gamma z)\right] + C$

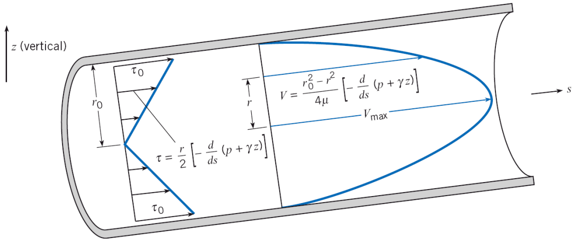
The velocity is 0 at the boundary,

One boundary condition:

$$V = 0 \quad \text{at} \quad r = r_0$$

$$\Rightarrow \quad \boxed{V = \frac{r_0^2 - r^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right]}$$

(parabolic profile)



Example 10.1:

Oil flows steadily in a vertical pipe. Pressure at z=100m is 200 kPa, and at z=85m it is 250 kPa.

Given: Diameter D = 3 cm
Viscosity μ = 0.5 Ns/m^2
Density ρ = 900 kg/m^3

Assume laminar flow.

Is the flow upward or downward? What is the velocity at the center and at r=6mm?

Example 10.1: Solution

First determine rate of change of p + γz

$$\frac{d}{ds}(p + \gamma z) = \frac{(p_{100} + \gamma z_{100}) - (p_{85} + \gamma z_{85})}{z_{100} - z_{85}}$$

$$= \frac{[200 \times 10^3 + 8830\ (100)] - [250 \times 10^3 + 8830\ (85)]}{15} = 5.53\ \mathrm{kN/m^3}$$

Since the velocity is given by

$$V = \frac{r_0^2 - r^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right]$$

the flow velocity is negative, i.e., downward.

The velocity at any point r is found from

$$V = \frac{r_0^2 - r^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right]$$

where we have already determined the value of

$$\frac{d}{ds}(p + \gamma z) = 5.53\ \mathrm{kN/m^3}$$

For r = 0, V = -0.622 m/s
For r = 6 mm, V = -0.522 m/s

Note that the velocity is in the direction of pressure <u>increase</u>. The flow direction is determined by the combination of pressure gradient and gravity. In this problem, the effect of gravity is stronger.

Head loss for laminar flow in a pipe

The mean velocity in the pipe is given by

$$\bar{V} = \frac{Q}{A} = \frac{1}{A}\int V\, dA$$

$$= \frac{1}{\pi r_0^2}\int_0^{r_0} \frac{r_0^2 - r^2}{4\mu}\left[-\frac{d}{ds}(p+\gamma z)\right](2\pi r\, dr)$$

$$= \frac{r_0^2}{8\mu}\left[-\frac{d}{ds}(p+\gamma z)\right]$$

$$= \frac{D^2}{32\mu}\left[-\frac{d}{ds}(p+\gamma z)\right]$$

Rearranging gives

$$\frac{d}{ds}(p+\gamma z) = -\frac{32\mu\bar{V}}{D^2}$$

which we integrate along s between sections 1 and 2:

$$p_2 - p_1 + \gamma(z_2 - z_1) = -\frac{32\mu\bar{V}}{D^2}(s_2 - s_1)$$

Identify the length of pipe section $L = s_2 - s_1$

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 + \frac{32\mu L\bar{V}}{\gamma D^2}$$

This is simply the energy equation for a pipe with head loss

$$\boxed{h_f = \frac{32\mu L\bar{V}}{\gamma D^2}}$$

Criterion for Turbulent vs. Laminar Flow in a Pipe

The behavior of flow in pipes is determined by the Reynolds number Re.

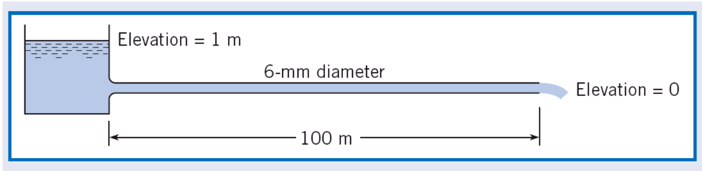
$$\boxed{Re = \frac{VD\rho}{\mu}}$$

Flow tends to become turbulent when Re > 3000.
Flow is always laminar when Re < 2000.

For 2000 < Re < 3000, the behavior is unpredictable and often switches back and forth between laminar and turbulent.

When conditions are carefully controlled so that the flow is perfectly motionless at the inlet of the pipe and the pipe is free of vibrations, then it is possible to maintain laminar flow even at Re > 3000.

Example 10.3: Determine rate of flow in the pipe



Fluid is kerosene with

Density $\rho = 820$ kg/m^3
Viscosity $\mu = 3.2 \times 10^{-3}$ Ns/m^2

We've solved this type of problem before…
The problem here is that we don't know (we are not told) whether or not the flow is laminar.

Example 10.3: Solution

We don't know the velocity, so we cannot compute the Reynolds number which tells us whether the flow is laminar or turbulent.

The pipe is quite thin, so we begin by assuming that the flow is laminar. Once we have the solution, we'll check whether that assumption was justified.

Energy equation (point 1 at surface of tank, point 2 at outlet):

$$\frac{p_1}{\gamma} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + \frac{32\mu L V_2}{\gamma D^2}$$

(If the flow were turbulent, we'd have to use a different form for the last term, the head loss).

$$\frac{p_1}{\gamma} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + \frac{32\mu L V_2}{\gamma D^2}$$

p_1, V_1, p_2 and z_2 are zero. We thus have all the information we need to solve for V_2

However, if the flow is laminar then the terms involving squares of velocity should be small, so we assume the term involving V_2^2 is zero (easier calculations…)

$$0 + 0 + 1\text{m} = 0 + 0 + 0 + \frac{32\mu L V_2}{\gamma D^2}$$

$$\Rightarrow V_2 = \frac{1\text{m} \times \gamma D^2}{32\mu L} = 28.2\text{mm/s}$$

This is our "guess" for the solution. Now we check whether our assumptions were justified.

$$Re = \frac{VD\rho}{\mu} = \frac{(0.0282\text{m/s})(0.006\text{m})^2(820\text{kg/m}^3)}{0.0032\text{Ns/m}^2} = 43.4$$

Re is much less than 2000, so the flow is laminar. That was our main assumption which is thus correct.

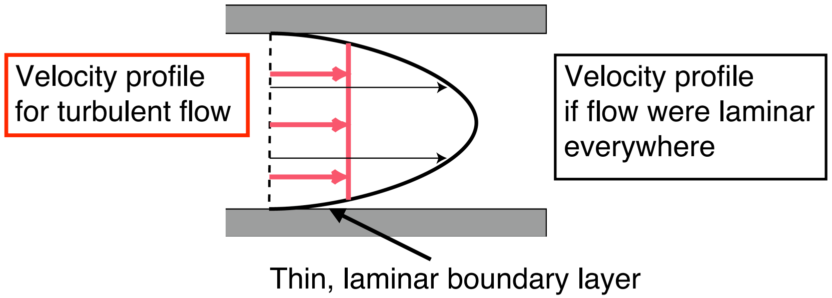
$$\frac{p_1}{\gamma} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + \frac{32\mu L V_2}{\gamma D^2}$$

We found that the 1st and 3rd circled terms = 1m.
We neglected the 2nd one.

$$\frac{\alpha_2 V_2^2}{2g} = \frac{V_2^2}{g} = 8.1 \times 10^{-5}\ \text{m}$$

This term is indeed negligible so our solution is OK.

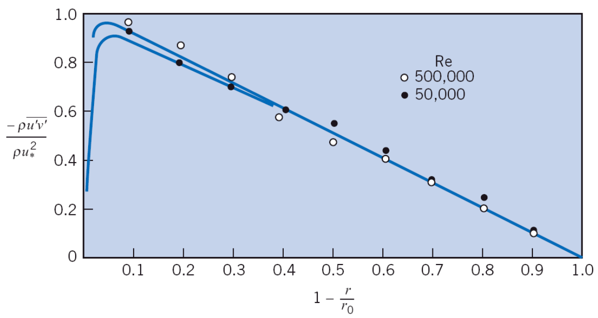
Turbulent flow is less efficient than laminar flow:



If flow could remain laminar, the pipe could transport more fluid for a given pressure gradient.

The swirls and eddies associated with turbulence make the fluid appear as though it had a much higher viscosity where flow is turbulent.

Same concept, different way of looking at it:



$$\tau_{\text{app}} = -\rho \overline{u'v'}$$

The effective mean stress (or apparent stress) is much greater than the stress expected for laminar flow.
Within the turbulent flow, this stress is approximately linear with radius.
The apparent stress depends on the turbulent velocity perturbations u' and v'.

Velocity distribution in smooth pipes:

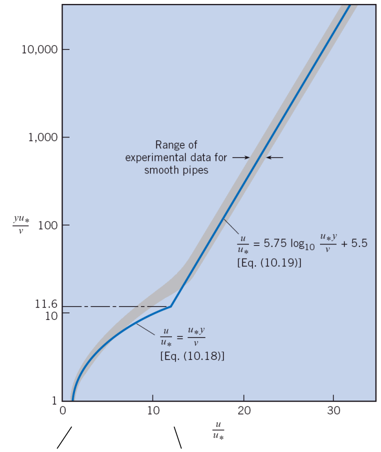
Experiments show:

$$\boxed{\frac{u}{u_*} = \frac{u_* y \rho}{\mu} \quad \text{for} \quad 0 < \frac{u_* y \rho}{\mu} < 5}$$

(laminar boundary layer)

$$\boxed{\frac{u}{u_*} = 5.75 \log \frac{u_* y \rho}{\mu} + 5.5 \quad \text{for} \quad 20 < \frac{u_* y \rho}{\mu} \leq 10^5}$$

where $u_* = \frac{\tau_0}{\rho}$



Laminar Turbulent

(note logarithmic scales)

More empirical (experimental) relations for smooth pipes:

$$\boxed{\tau_0 = \frac{f \rho V^2}{8}}$$

shear stress at wall

$$\boxed{h_f = f \frac{L V^2}{2gD}}$$

head loss (Darcy-Weisbach equation)

where $f = \frac{64}{Re}$ for laminar flow

$$\frac{1}{\sqrt{f}} = 2 \log(Re\sqrt{f}) - 0.8$$

For turbulent flow with Re > 3000

Rough Pipes

Velocity distribution

$$\frac{u}{u_*} = 5.75 \log \frac{y}{k} + B$$

k is a parameter that characterizes the height of the roughness elements.

B is a parameter that is a function of the type, concentration, and the size variation of the roughness.

y is distance from wall.

Rough Pipes

$$\left(\frac{k_s}{D}\right) Re < 10$$

Low Reynolds number or small roughness elements:
Roughness unimportant, pipe considered smooth

$$\left(\frac{k_s}{D}\right) Re > 1000$$

High Reynolds number or large roughness elements:
Fully rough, f independent of Reynolds number.

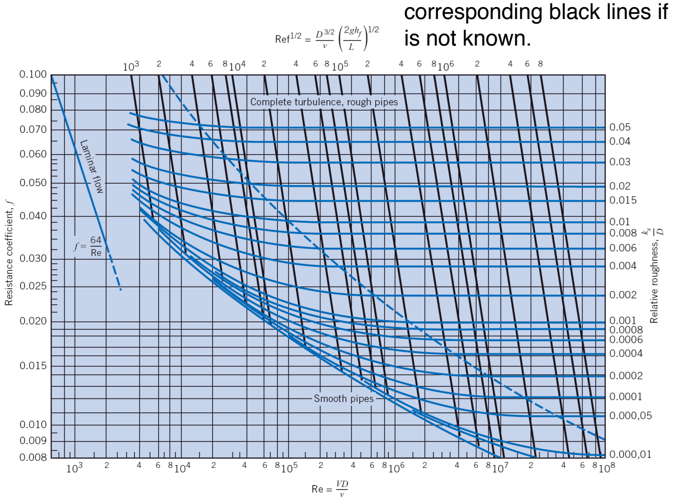
$$\tau_0 = \frac{f\,\rho\, V^2}{8} \quad \text{and} \quad h_f = f\frac{L\, V^2}{2gD} \quad \text{are still valid}$$

TABLE 10.2 EQUIVALENT SAND GRAIN ROUGHNESS, k_s, FOR VARIOUS PIPE MATERIALS		
Boundary Material	**k_s, millimeters**	**k_s, inches**
Glass, plastic	Smooth	Smooth
Copper or brass tubing	0.0015	6×10^{-5}
Wrought iron, steel	0.046	0.002
Asphalted cast iron	0.12	0.005
Galvanized iron	0.15	0.006
Cast iron	0.26	0.010
Concrete	0.3 to 3.0	0.012–0.12
Riveted steel	0.9–9	0.035–0.35
Rubber pipe (straight)	0.025	0.001

How to find f for rough pipes? Moody diagram:

use this parameter and the corresponding black lines if velocity is not known.

f; the value we're looking for



Get value for ks from table; each value of ks/D corresponds to one of the blue curves

Reynolds number (if velocity is known)

Example 10.4: Find head loss per kilometer of pipe.

Pipe is a 20-cm asphalted cast-iron pipe.
Fluid is water.
Flow rate is $Q = 0.05\ m^3/s$.

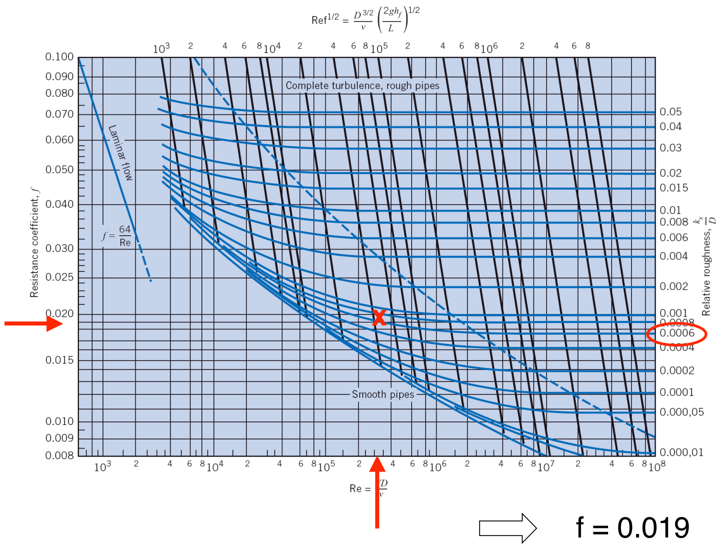
Solution:

First compute Reynolds number

$$Re = \frac{VL\rho}{\mu} = \frac{QL\rho}{A\mu} = 3.18 \times 10^5$$

From Table 10.2, k_s = 0.12 mm for asphalted cast-iron pipe.

So, $k_s/D = 0.0006$



$\Rightarrow$ f = 0.019

With f = 0.019, we get the head loss h_f from the Darcy-Weisbach equation:

$$h_f = f\frac{LV^2}{2Dg} = 0.0019\left(\frac{1000\text{m}}{0.20\text{m}}\right)\left(\frac{(1.59\text{m/s})^2}{2(9.81\text{m/s}^2)}\right) = 12.2\text{m}$$

Example 10.5: Find volume flow rate Q.

Similar to last problem:

Pipe is 20-cm asphalted cast-iron.
Fluid is water.
Head loss per kilometer is 12.2 m.

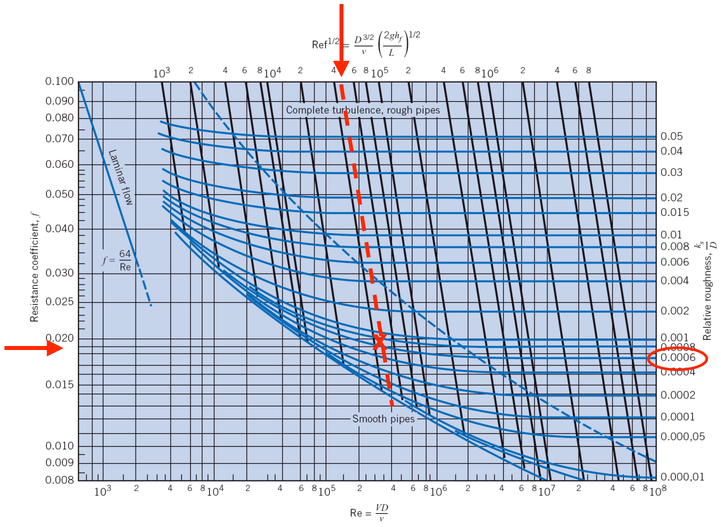
The difference to the previous problem is that we don't know the velocity, so we can't compute Re.

Compute instead

$$D^{3/2}\frac{\sqrt{gh_f/L}}{\nu} = 4.38 \times 10^4$$

where $\nu = \dfrac{\mu}{\rho}$

is the kinematic viscosity.



Relative roughness same as previous problem

again $\Rightarrow$ f = 0.019

Now we use the Darcy-Weisbach equation again to get V

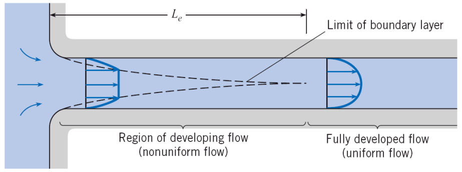
$$h_f = f\frac{LV^2}{2Dg}$$

$$\Rightarrow \qquad V = \sqrt{\frac{2Dgh_f}{fL}}$$

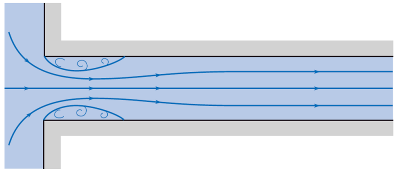
$$V = 1.59\text{m/s}$$

$$Q = A\ V = \frac{\pi D^2}{4}V = 0.050\text{m}^3/\text{s}$$

Flow at pipe inlets and losses from fittings



Rounded inlet



Sharp-edged inlet

Head loss for inlets, outlets, and fittings:

$$h_L = K\frac{V^2}{2g}$$

where K is a parameter that depends on the geometry. For a well-rounded inlet, K = 0.1, for abrupt inlet K = 0.5 (much less resistance for rounded inlet).

Bends in pipes:

Sharp bends result in separation downstream of the bend.

The turbulence in the separation zone causes flow resistance.

Greater radius of bend reduces flow resistance.

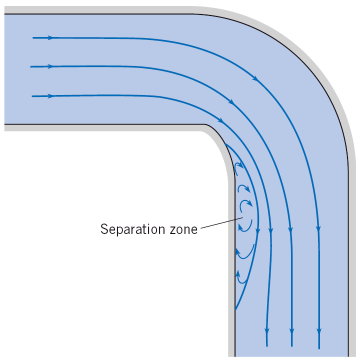
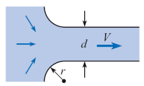
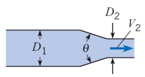
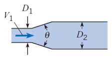
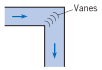
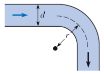


TABLE 10.3 LOSS COEFFICIENTS FOR VARIOUS TRANSITIONS AND FITTINGS

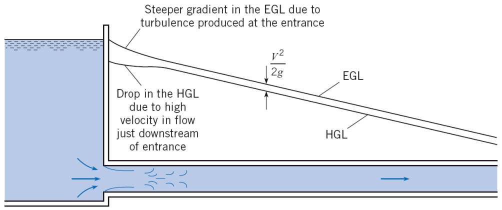
Description	Sketch	Additional Data		K	Source
Pipe entrance		r/d		K_e	(18)†
		0.0		0.50	
		0.1		0.12	
$h_L = K_e V^2/2g$		>0.2		0.03	
Contraction		D_2/D_1	K_C $u = 60°$	K_C $u = 180°$	(18)
		0.0	0.08	0.50	
		0.20	0.08	0.49	
		0.40	0.07	0.42	
		0.60	0.06	0.27	
		0.80	0.06	0.20	
$h_L = K_C V_2^2/2g$		0.90	0.06	0.10	

* Engineering handbooks usually include extensive tables of loss coefficients. References (18), (19), (20), (21), (22), and (23) are particularly useful in this respect.

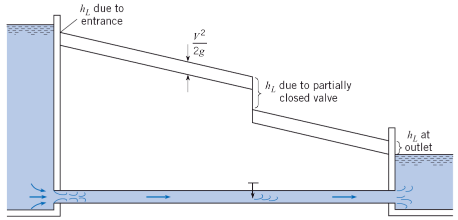
TABLE 10.3 LOSS COEFFICIENTS FOR VARIOUS TRANSITIONS AND FITTINGS (CONTINUED)					
Description	**Sketch**	**Additional Data**		***K***	**Source**
Expansion		D_1/D_2	K_E $u = 20°$	K_E $u = 180°$	(17)
		0.0		1.00	
		0.20	0.30	0.87	
		0.40	0.25	0.70	
		0.60	0.15	0.41	
$h_L = K_E V_1^2/2g$		0.80	0.10	0.15	
90° miter bend		Without vanes		$K_b = 1.1$	(23)
		With vanes		$K_b = 0.2$	(23)
90° smooth bend		r/d			(24) and (17)
		1	$K_b = 0.35$		
		2	0.19		
		4	0.16		
		6	0.21		
		8	0.28		
		10	0.32		
Threaded pipe fittings	Globe valve—wide open	$K_v = 10.0$			(23)
	Angle valve—wide open	$K_v = 5.0$			
	Gate valve—wide open	$K_v = 0.2$			
	Gate valve—half open	$K_v = 5.6$			
	Return bend	$K_b = 2.2$			
	Tee				
	straight-through flow	$K_t = 0.4$			
	side-outlet flow	$K_t = 1.8$			
	90° elbow	$K_b = 0.9$			
	45° elbow	$K_b = 0.4$			

†Reprinted by permission of the American Society of Heating, Refrigerating and Air Conditioning Engineers, Atlanta, Georgia, from the 1981 *ASHRAE Handbook—Fundamentals*.

Transition losses and grade lines



Head loss due to transitions (inlets, etc.) is distributed over some distance.
Details are often quite complicated.



Approximation: Abrupt losses at a point.

Turbulent Flow in Non-Circular Conduits

Relations for shear stress at boundary and for head loss are similar to those for circular conduits:

Circular pipes

$$\tau_0 = \frac{D}{4}\left[-\frac{d}{ds}(p + \gamma z)\right]$$

$$h_f = f\frac{LV^2}{2Dg}$$ (Darcy-Weisbach equation)

Non-circular conduits

$$\tau_0 = \frac{A}{P}\left[-\frac{d}{ds}(p + \gamma z)\right]$$

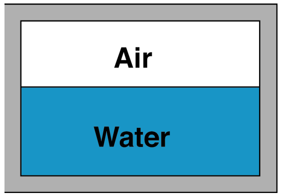
$$h_f = f\frac{P}{4A}\frac{LV^2}{2g}$$

here A is cross sectional area and P is perimeter of pipe.

In these equations, the circular pipe diameter D was simply replaced by 4 A / P.

Hydraulic radius: $$R_h = \frac{A}{P}$$

The conduit need not be filled with fluid:



Cross section of rectangular conduit.

A is the cross-sectional area of the pipe

P is the wetted perimeter of the pipe, that is, the length of pipe perimeter that is in contact with the fluid.

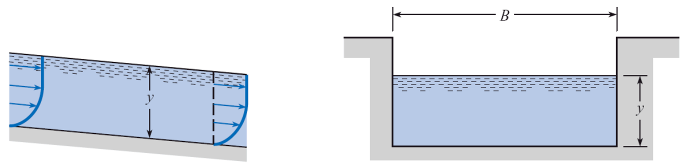
Flow problems for non-circular conduits can be solved the same way as problems for circular pipes.

Simply replace D by $4R_h$

Relative roughness is $\dfrac{k_s}{4R_h}$

Reynolds number is $Re = \dfrac{4VR_h\rho}{\mu}$

Uniform free-surface flows



Side view   End view

Same equations as for non-circular conduits.

However, A is only the cross-sectional area of the fluid.

As for pipes, is laminar for $\dfrac{4VR_h\rho}{\mu} < 2000$

and turbulent for $\dfrac{4VR_h\rho}{\mu} > 3000$

(But for some reason the Reynolds number for open channels is usually defined as $Re = \dfrac{VR_h\rho}{\mu}$)

Chezy and Manning Equations (for open channels)

Start with head-loss equation: $h_f = \dfrac{fLV^2}{4R_h\,2g}$

In an open channel, the hydraulic grade line is the same as the free surface, so that the slope is given by

$$S_0 = \frac{h_f}{L}$$

and hence $R_h\,S_0 = \dfrac{f}{8g}V^2$

$$\left.\begin{aligned} V &= C\sqrt{R_h S_0} \\ Q &= CA\sqrt{R_h S_0} \end{aligned}\right\} \quad \text{with} \quad C = \sqrt{\frac{8g}{f}}$$

(Chezy equation)

Thus far, we have only re-organized the formulas we used before.

However, the way C is commonly determined in the Chezy equation is

$$C = \frac{R_h^{1/6}}{n}$$

where n is a resistance coefficient called Manning's n.

TABLE 10.4 TYPICAL VALUES OF THE ROUGHNESS COEFFICIENT n	
Lined Canals	n
Cement plaster	0.011
Untreated gunite	0.016
Wood, planed	0.012
Wood, unplaned	0.013
Concrete, troweled	0.012
Concrete, wood forms, unfinished	0.015
Rubble in cement	0.020
Asphalt, smooth	0.013
Asphalt, rough	0.016
Corrugated metal	0.024
Unlined Canals	
Earth, straight and uniform	0.023
Earth, winding and weedy banks	0.035
Cut in rock, straight and uniform	0.030
Cut in rock, jagged and irregular	0.045
Natural Channels	
Gravel beds, straight	0.025
Gravel beds plus large boulders	0.040
Earth, straight, with some grass	0.026
Earth, winding, no vegetation	0.030
Earth, winding, weedy banks	0.050
Earth, very weedy and overgrown	0.080

Recall in the previous approach we used the Moody diagram (that complicated graph).

In the Moody diagram, we used the relative roughness, k_s / D.

Here, there is only one type of roughness which is independent of the channel size.

The approach we used before is more accurate. However, the Chezy equation is still commonly used.

An additional word of caution:

Substituting for C, the Chezy equation can be written as

$$Q = \frac{1.0}{n} A R_h^{2/3} S_0^{1/2}$$

It is valid only in SI units.

For “traditional units” (feet, pounds, …) the equation is

$$Q = \frac{1.49}{n} A R_h^{2/3} S_0^{1/2} \quad \text{(Manning's equation)}$$

(This sort of stuff only happens if you leave out the proper units somewhere; e.g., using a unitless parameter instead of keeping the units it should have. This is highly unscientific!)

Best Hydraulic Section

From Chezy formula: $Q = \frac{1.0}{n} A R_h^{2/3} S_0^{1/2}$

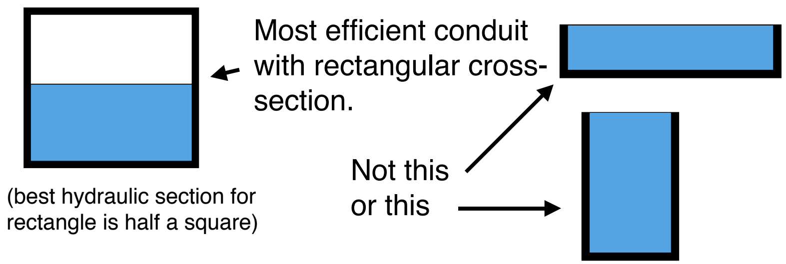
for a given slope S_0, the flow rate is proportional to

$$Q \propto A \left(\frac{A}{P}\right)^{\frac{2}{3}}$$

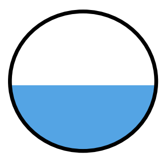
Large cross-sectional area A gives high Q.

Large wetted perimeter P gives low Q.

Highest flow rate Q for certain types of shapes



Best rounded shape:
Half of a circle.



Best trapezoid:
Half of a hexagon.