Cosmic Chemistry: Cosmogony

Mathematical Models

STUDENT ACTIVITY

Models in science come in different forms. A physical model that you probably are familiar with is an anatomically detailed model of the human body. Mathematical models are less commonly found in science classes, but they form the core of modern cosmology. Mathematical models are extremely powerful because they usually enable predictions to be made about a system. The predictions then provide a road map for further experimentation. Consequently, it is important for you to develop an appreciation for this type of model as you learn more about cosmology.

Two sections of the activity develop mathematical models of direct relevance to cosmology and astronomy. The math skills required in the activity increase with each section, but nothing terribly advanced is required.

A very common approach to the mathematical modeling of a physical system is to collect a set of experimental data and then figure out a way to graph the data so that one gets a straight line. Once a straight line is obtained, it is possible to generalize the information contained in the straight line in terms of the powerful algebraic equation:

$$y = mx + b$$

You probably are familiar with this equation. In it y represents a value on the y axis, x represents a value on the x axis, m represents the slope of the straight line, and b represents the value of the intercept of the line on the y axis. In all sections of this activity, your goal will be to analyze and then graph a set of data so that you obtain a straight line. Then you will derive the equation that describes the line, and use the equation to make predictions about the system. So relax and have fun with math!

PART 1

a) In this part of the activity, your team will investigate the way in which points on an elastic material move as the material is stretched. Discuss among yourselves the experimental set-up that you will use. Your discussion should include:
    i.   the number of points that you wish to follow;
    ii. how you will mark the points;
    iii. how far you will stretch the elastic; and
    iv. how you will hold the elastic in the stretched state while you make measurements on the positions of the points.
    Furthermore, you should decide on what point will be your "reference point" from which all measurements are made (i.e., the "zero" point on the elastic). Also discuss the meaning of "rate" or "speed." Everyone on the team should understand that rate is distance traveled divided by the time of travel. Discuss whether or not you actually want to measure the actual time taken to stretch the elastic material or whether you will treat this as a constant that applies equally to all points as they move. If you keep stretching time constant, then its exact value does not come into play at all.

b) As a team, develop a diagram showing how you think the positions of a set of points on an elastic material will change as the material is stretched. For example, if you have points located at 4, 7, and 15 cm from the end of the elastic, show where you think these points will be located (relative to the end) when the material is stretched 10 cm.

c) Write a description of the procedure that your team will employ, including materials needed before submitting it to your teacher. Include your diagram.

d) After your teacher approves your plans and diagram, acquire the materials you need and conduct the experiment. Record your measurements.
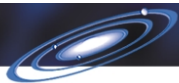
e) Construct a graph that shows the rate of movement of the points on the y-axis and the initial positions of the points with respect to the reference point on the x-axis. Be sure to construct a graph that utilizes most of the graph paper.

f) Draw a best-fit line through the points on the graph. If you made careful measurements, it will be linear, (i.e., a straight line) and it will pass through the origin of the graph.

g) Noting that the y intercept has a value of zero, the equation for a straight line becomes in this case:

$$y = mx + 0$$

Select two points on the line and from their x- and y-coordinates, and determine the slope of the line.

h) Submit a written report to your teacher in which you discuss whether the prediction made in step b) was a good one in light of the experimental evidence that you now have. Include in your report your graph and your derived equation. Also, include a calculation of the rate of movement of a point initially located 200 cm from the initial reference point, assuming that you have a very long piece of elastic. **Last, include in your report a statement of how this experiment relates to the expansion of the universe.** Refer to Appendix A, "Cosmology" for help in writing this statement.

PART 2

In all three parts of this activity, you will develop mathematical models. The first two parts deal with hypothetical situations, but the last part pertains to actual planetary motion. In each case you will be presented with a set of data that you will need to analyze and then graph in such a way as to produce a straight line. From the straight line, you will derive an equation that models the original data.
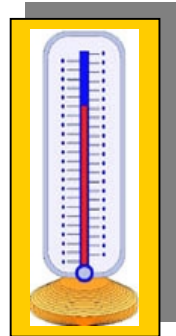
a) In this section, let us imagine that there is another universe—call it HOHUM—where the physical laws are similar to ours in many respects, but different in others. We wish to model one of the distinctly different aspects of HOHUM, namely that the mass of an object depends on its temperature. Yes, this is weird. But remember, we are talking about another universe. Who is to say that this might not be possible?

For fun, let's assume that the data shown below are available, where q represents the mass of a HOHUM object in grams and T represents temperature in degrees Celsius.

Temperature, T	10	20	30	40	50
Mass, q	7	26	69	148	275

Lets say that a HOHUM scientist communicates to you the idea that the general relationship between q and T is $q = (HO)T + (HUM)T^3$, where HO and HUM are constants. Obviously, for this equation to be very useful, you need to know the values of the constants. To obtain these values, remember that when you plot data to get a straight line of general form, $y = mx + b$, the constants m and b can be evaluated from the slope and intercept of the straight line.

Since b is a constant, you probably will want to rearrange your initial equation to provide a similar constant term, HO, on the right-hand side, that is: $q/T = (HUM)T^2 + (HO)$. Now it should be clear that the rearranged equation is in the form of an equation for a straight line, and if we plot q/T versus T^2, the straight line obtained will have a slope of HUM and an intercept of HO. In other words, q/T is the y and T^2 is the x of the generalized equation.

Draw a graph based on the data above and determine the value of the constants HUM and HO. Now, write your final equation on the graph paper showing how mass, q, depends on T in the HOHUM universe. This is your mathematical model of the relationship of mass to temperature in the strange universe HOHUM. Finally, predict the mass of an object having a temperature of 100 degrees Celsius and write your prediction on the graph paper. Turn your graph in to the teacher.
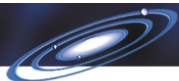
b) In this section, you will attack a problem that deals with the variation of atmospheric pressure with altitude on a planet. Let us assume that a spacecraft has arrived at a distant planet and that a probe dropped through the atmosphere has sent back the following data to you:

Altitude, km	0	2	5	8	10	15	20	25	30
Pressure, ktorr	3160	1910	890	417	251	70.8	12.0	5.60	1.60

A torr is a unit of pressure equivalent to 1 mm of Hg. One atmosphere of pressure is the same as 760 torr.

Again, the objective is to find a mathematical expression that models the variation of the pressure with the altitude. As before, the technique is to find some way to graph the data to obtain a straight line.
i. Construct a graph of pressure, P, (y-axis) versus altitude A, (x-axis). Discuss among classmates why the graph shows that the equation sought is NOT of the form $P = mA + B$.
ii. Notice that the pressure rises very rapidly with decreasing altitude. This suggests that a fruitful approach might be to involve logarithms, since logarithms (logs) "compress" large numbers into smaller ones. For example, the log of 100 is 2, the log of 1000 is 3, and so on. So, junior spaceperson, how about trying a graph of log P versus altitude? Go ahead and construct such a graph.
iii. Did you obtain a straight line? You should have. Now you are in position to develop a mathematical statement that models the pressure variation as a function of altitude. Note the arithmetic sign of the slope carefully.
iv. Based on the equation you have developed, you are now in a position to predict the altitude at which the pressure would be 100 ktorr. On the one that is a straight line, show your equation and your prediction of the altitude where the pressure is 100 ktorr. Turn in both of your graphs.

c) The following scenario involves actual data from the solar system, as opposed to the previous two hypothetical situations. Shown in the table below are the distances of the planets from the sun and the time it takes the planets to travel around the sun one time (the sidereal period).

Planet	Distance, D (millions of km)	Sidereal Period, T (Days)
Mercury	57.9	88
Venus	108.2	225
Earth	149.6	365
Mars	227.9	687
Jupiter	778.3	4329
Saturn	1427	10753
Uranus	2870	30660
Neptune	4497	60150
Pluto	5914	90670

Construct three graphs, one of D versus T, one of log T versus D, and one of log D versus log T. From these graphs decide on the form of the mathematical model that relates D and T, and then derive the equation that expresses the relationship between sidereal period and distance.

Finally, if your teacher instructs you to do so, show that your result is consistent with Kepler's third law. This seventeenth century law states that the square of the sidereal period is proportional to the cube of the distance of the planet from the sun.

Turn your three graphs, your derived mathematical model, and, if so instructed, your demonstration that you have derived Kepler's third law in to your teacher.