Sensitivity Analysis

3

We have already been introduced to sensitivity analysis in Chapter 1 via the geometry of a simple example. We saw that the values of the decision variables and those of the slack and surplus variables remain unchanged even though some coefficients in the objective function are varied. We also saw that varying the righthand-side value for a particular constraint alters the optimal value of the objective function in a way that allows us to impute a per-unit value, or *shadow price*, to that constraint. These shadow prices and the shadow prices on the implicit nonnegativity constraints, called *reduced costs*, remain unchanged even though some of the righthand-side values are varied. Since there is always some uncertainty in the data, it is useful to know over what range and under what conditions the components of a particular solution remain unchanged. Further, the sensitivity of a solution to changes in the data gives us insight into possible technological improvements in the process being modeled. For instance, it might be that the available resources are not balanced properly and the primary issue is not to resolve the most effective allocation of these resources, but to investigate what additional resources should be acquired to eliminate possible bottlenecks. Sensitivity analysis provides an invaluable tool for addressing such issues.

There are a number of questions that could be asked concerning the sensitivity of an optimal solution to changes in the data. In this chapter we will address those that can be answered most easily. Every commercial linear-programming system provides this elementary sensitivity analysis, since the calculations are easy to perform using the tableau associated with an optimal solution. There are two variations in the data that invariably are reported: objective function and righthand-side ranges. The objective-function ranges refer to the range over which an individual coefficient of the objective function can vary, without changing the basis associated with an optimal solution. In essence, these are the ranges on the objective-function coefficients over which we can be sure the values of the decision variables in an optimal solution will remain unchanged. The righthand-side ranges refer to the range over which an individual righthand-side value can vary, again without changing the basis associated with an optimal solution. These are the ranges on the righthand-side values over which we can be sure the values of the shadow prices and reduced costs will remain unchanged. Further, associated with each range is information concerning how the basis would change if the range were *exceeded*. These concepts will become clear if we deal with a specific example.

3.1 AN EXAMPLE FOR ANALYSIS

We will consider for concreteness the custom-molder example from Chapter 1; in order to increase the complexity somewhat, let us add a third alternative to the production possibilities. Suppose that, besides the six-ounce juice glasses x_1 and the ten-ounce cocktail glasses x_2, our molder is approached by a new customer to produce a champagne glass. The champagne glass is not difficult to produce except that it must be molded in two separate pieces—the bowl with stem and then base. As a result, the production time for the champagne glass is 8 hours per hundred cases, which is greater than either of the other products. The storage space required for the champagne glasses is 1000 cubic feet per hundred cases; and the contribution is $6.00

per case, which is higher than either of the other products. There is no limit on the demand for champagne glasses. Now what is the optimal product mix among the three alternatives?

The formulation of the custom-molding example, including the new activity of producing champagne glasses, is straightforward. We have exactly the same capacity limitations—hours of production capacity, cubic feet of warehouse capacity, and limit on six-ounce juice-glass demand—and one additional decision variable for the production of champagne glasses. Letting

$$x_1 = \text{Number of cases of six-ounce juice glasses, in hundreds;}$$
$$x_2 = \text{Number of cases of ten-ounce cocktail glasses, in hundreds;}$$
$$x_3 = \text{Number of cases of champagne glasses, in hundreds;}$$

and measuring the contribution in hundreds of dollars, we have the following formulation of our custom-molder example:

$$\text{Maximize } z = 5x_1 + 4.5x_2 + 6x_3, \qquad \text{(hundreds of dollars)}$$

subject to:

$$
\begin{aligned}
6x_1 + 5x_2 + 8x_3 &\le 60, & &\text{(production capacity; hours)} \\
10x_1 + 20x_2 + 10x_3 &\le 150, & &\text{(warehouse capacity; hundreds of sq. ft.)} \\
x_1 &\le 8, & &\text{(demand for 6 oz. glasses; hundreds of cases)}
\end{aligned}
\tag{1}
$$

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

If we add one slack variable in each of the less-than-or-equal-to constraints, the problem will be in the following canonical form for performing the simplex method:

$$
\begin{aligned}
6x_1 + 5x_2 + 8x_3 + x_4 &&&= 60, & (2) \\
10x_1 + 20x_2 + 10x_3 &+ x_5 &&= 150, & (3) \\
x_1 + && + x_6 &= 8, & (4) \\
5x_1 + 4.5x_2 + 6x_3 && - z &= 0. & (5)
\end{aligned}
$$

The corresponding initial tableau is shown in Tableau 1.* After applying the simplex method as described

Tableau 1

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_4	60	6	5	8	1		
x_5	150	10	20	10		1	
x_6	8	1	0	0			1
$(-z)$	0	5	4.5	6			

in Chapter 2, we obtain the final tableau shown in Tableau 2.*

Since the final tableau is in canonical form and all objective-function coefficients of the nonbasic variables are currently nonpositive, we know from Chapter 2 that we have the optimal solution, consisting of $x_1 = 6\frac{3}{7}$, $x_2 = 4\frac{2}{7}$, $x_6 = 1\frac{4}{7}$, and $z = 51\frac{3}{7}$.

In this chapter we present results that depend only on the initial and final tableaus of the problem. Specifically, we wish to analyze the effect on the optimal solution of changing various elements of the problem data without re-solving the linear program or having to remember any of the intermediate tableaus

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* Excel spreadsheet available at `http://web.mit.edu/15.053/www/Sect3.1_Tableaus.xls`

Tableau 2

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_2	$4\frac{2}{7}$		1	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{3}{35}$	
x_6	$1\frac{4}{7}$			$-\frac{11}{7}$	$-\frac{2}{7}$	$\frac{1}{14}$	1
x_1	$6\frac{3}{7}$	1		$\frac{11}{7}$	$\frac{2}{7}$	$-\frac{1}{14}$	
$(-z)$	$-51\frac{3}{7}$			$-\frac{4}{7}$	$-\frac{11}{14}$	$-\frac{1}{35}$	

generated in solving the problem by the simplex method. The type of results that can be derived in this way are conservative, in the sense that they provide sensitivity analysis for changes in the problem data small enough so that the same decision variables remain basic, but not for larger changes in the data. The example presented in this section will be used to motivate the discussions of sensitivity analysis throughout this chapter.

3.2 SHADOW PRICES, REDUCED COSTS, AND NEW ACTIVITIES

In our new variation of the custom-molder example, we note that the new activity of producing champagne glasses is not undertaken at all. An immediate question arises, could we have known this without performing the simplex method on the entire problem? It turns out that a proper interpretation of the shadow prices in the Chapter 1 version of the problem would have told us that producing champagne glasses would not be economically attractive. However, let us proceed more slowly. Recall the definition of the shadow price associated with a particular constraint.

> ***Definition.***   The *shadow price* associated with a particular constraint is the change in the optimal value of the objective function per unit increase in the righthand-side value for that constraint, all other problem data remaining unchanged.

In Chapter 1 we implied that the shadow prices were readily available when a linear program is solved. Is it then possible to determine the shadow prices from the final tableau easily? The answer is yes, in general, but let us consider our example for concreteness.

Suppose that the production capacity in the first constraint of our model

$$6x_1 + 5x_2 + 8x_3 + x_4 = 60 \tag{6}$$

is increased from 60 to 61 hours. We then essentially are procuring one additional unit of production capacity at no cost. We can obtain the same result algebraically by allowing the slack variable x_4 to take on negative values. If x_4 is replaced by $x_4 - 1$ (i.e., from its optimal value $x_4 = 0$ to $x_4 = -1$), Eq.(6) becomes:

$$6x_1 + 5x_2 + 8x_3 + x_4 = 61,$$

which is exactly what we intended.

Since x_4 is a slack variable, it does not appear in any other constraint of the original model formulation, nor does it appear in the objective function. Therefore, this replacement does not alter any other righthand-side value in the original problem formulation. What is the contribution to the optimal profit of this additional unit of capacity? We can resolve this question by looking at the objective function of the final tableau, which is given by:

$$z = 0x_1 + 0x_2 - \tfrac{4}{7}x_3 - \tfrac{11}{14}x_4 - \tfrac{1}{35}x_5 + 0x_6 + 51\tfrac{3}{7}. \tag{7}$$

The optimality conditions of the simplex method imply that the optimal solution is determined by setting the nonbasic variables $x_3 = x_4 = x_5 = 0$, which results in a profit of $51\frac{3}{7}$. Now, if we are allowed to make $x_4 = -1$, the profit increases by $\frac{11}{14}$ hundred dollars for each additional unit of capacity available. This, then, is the marginal value, or shadow price, for production hours.

The righthand side for every constraint can be analyzed in this way, so that the shadow price for a particular constraint is merely the negative of the coefficient of the appropriate slack (or artificial) variable in the objective function of the final tableau. For our example, the shadow prices are $\frac{11}{14}$ hundred dollars per hour of production capacity, $\frac{1}{35}$ hundred dollars per hundred cubic feet of storage capacity, and zero for the limit on six-ounce juice-glass demand. It should be understood that the shadow prices are associated with the constraints of the problem and not the variables. They are in fact the marginal worth of an additional unit of a particular righthand-side value.

So far, we have discussed shadow prices for the explicit structural constraints of the linear-programming model. The nonnegativity constraints also have a shadow price, which, in linear-programming terminology, is given the special name of reduced cost.

> ***Definition.***   The *reduced cost* associated with the nonnegativity constraint for each variable is the shadow price of that constraint (i.e., the corresponding change in the objective function per unit increase in the lower bound of the variable).

The reduced costs can also be obtained directly from the objective equation in the final tableau. In our example, the final objective form is

$$z = 0x_1 + 0x_2 - \tfrac{4}{7}x_3 - \tfrac{11}{14}x_4 - \tfrac{1}{35}x_5 + 0x_6 + 51\tfrac{3}{7}. \tag{8}$$

Increasing the righthand side of $x_3 \geq 0$ by one unit to $x_3 \geq 1$ forces champagne glasses to be used in the final solution. From (8), the optimal profit decreases by $-\frac{4}{7}$. Since the basic variables have values $x_1 = 6\frac{3}{7}$ and $x_2 = 4\frac{2}{7}$, increasing the righthand sides of $x_1 \geq 0$ and $x_2 \geq 0$ by a small amount does not affect the optimal solution, so their reduced costs are zero. Consequently, in every case, the shadow price for the nonnegativity constraint on a variable is the objective coefficient for this variable in the final canonical form. For basic variables, these reduced costs are zero.

Alternatively, the reduced costs for all decision variables can be computed directly from the shadow prices on the structural constraints and the objective-function coefficients. In this view, the shadow prices are thought of as the opportunity costs associated with diverting resources away from the optimal production mix. For example, consider x_3. Since the new activity of producing champagne glasses requires 8 hours of production capacity per hundred cases, whose opportunity cost is $\frac{11}{14}$ hundred dollars per hour, and 10 hundred cubic feet of storage capacity per hundred cases, whose opportunity cost is $\frac{1}{35}$ hundred dollars per hundred cubic feet, the resulting total opportunity cost of producing one hundred cases of champagne glasses is:

$$\left(\tfrac{11}{14}\right)8 + \left(\tfrac{1}{35}\right)10 = \tfrac{46}{7} = 6\tfrac{4}{7}.$$

Now the contribution per hundred cases is only 6 hundred dollars so that producing any champagne glasses is not as attractive as producing the current levels of six-ounce juice glasses and ten-ounce cocktail glasses. In fact, if resources were diverted from the current optimal production mix to produce champagne glasses, the optimal value of the objective function would be reduced by $\frac{4}{7}$ hundred dollars per hundred cases of champagne glasses produced. This is exactly the reduced cost associated with variable x_3. This operation of determining the reduced cost of an activity from the shadow price and the objective function is generally referred to as *pricing out an activity*.

Given the reduced costs, it becomes natural to ask how much the contribution of the new activity would have to increase to make producing champagne glasses attractive? Using the opportunity-cost interpretation, the contribution clearly would have to be $\$6\frac{4}{7}$ in order for the custom-molder to be indifferent to transferring resources to the production of champagne glasses. Since the reduced cost associated with the new activity $6 - 6\frac{4}{7} = -\frac{4}{7}$ is negative, the new activity will not be introduced into the basis. If the reduced cost had been positive, the new activity would have been an attractive candidate to introduce into the basis.

The shadow prices determined for the Chapter 1 version of the custom-molder example are the same as those determined here, since the optimal solution is unchanged by the introduction of the new activity

of producing champagne glasses. Had the new activity been priced out at the outset, using the shadow prices determined in Chapter 1, we would have immediately discovered that the opportunity cost of diverting resources from the current solution to the new activity exceeded its potential contribution. There would have been no need to consider the new activity further. This is an important observation, since it implies that the shadow prices provide a mechanism for screening new activities that were not included in the initial model formulation. In a maximization problem, if any new activity prices out negatively using the shadow prices associated with an optimal solution, it may be immediately dropped from consideration. If, however, a new activity prices out positively with these shadow prices, it must be included in the problem formulation and the new optimal solution determined by pivoting.

General Discussion

The concepts presented in the context of the custom-molder example can be applied to any linear program. Consider a problem in initial canonical form:

$$
\begin{array}{llll}
& & & \begin{array}{c} \textit{Shadow} \\ \textit{price} \end{array} \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} & & = b_1 & y_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + x_{n+2} & = b_2 & y_2 \\
\vdots & & \vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + & \cdots + x_{n+m} & = b_m & y_m \\
\hline
(-z) + c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \cdots + 0x_{n+m} & = 0 &
\end{array}
$$

The variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ are either slack variables or artificial variables that have been introduced in order to transform the problem into canonical form.

Assume that the optimal solution to this problem has been found and the corresponding final form of the objective function is:

$$
\begin{aligned}
(-z) + \bar{c}_1x_1 + \bar{c}_2x_2 + \cdots + \bar{c}_nx_n + \bar{c}_{n+1}x_{n+1} & \\
+ \bar{c}_{n+2}x_{n+2} + \cdots + \bar{c}_{n+m}x_{n+m} & = -\bar{z}_0.
\end{aligned} \tag{9}
$$

As we have indicated before, $\bar{c}_j$ is the reduced cost associated with variable x_j. Since (9) is in canonical form, $\bar{c}_j = 0$ if x_j is a basic variable. Let y_i denote the shadow price for the ith constraint. The arguments from the example problem show that the negative of the final objective coefficient of the variable x_{n+i} corresponds to the shadow price associated with the ith constraint. Therefore:

$$
\bar{c}_{n+1} = -y_1, \qquad \bar{c}_{n+2} = -y_2, \qquad \ldots, \qquad \bar{c}_{n+m} = -y_m. \tag{10}
$$

Note that this result applies whether the variable x_{n+i} is a slack variable (i.e., the ith constraint is a less-than-or-equal-to constraint), or whether x_{n+i} is an artificial variable (i.e., the ith constraint is either an equality or a greater-than-or-equal-to constraint).

We now shall establish a fundamental relationship between shadow prices, reduced costs, and the problem data. Recall that, at each iteration of the simplex method, the objective function is transformed by subtracting from it a multiple of the row in which the pivot was performed. Consequently, the final form of the objective function could be obtained by subtracting multiples of the original constraints from the original objective function. Consider first the final objective coefficients associated with the original basic variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$. Let $\pi_1, \pi_2, \ldots, \pi_n$ be the multiples of each row that are subtracted from the original objective function to obtain its final form (9). Since x_{n+i} appears only in the ith constraint and has a $+1$ coefficient, we should have:

$$
\bar{c}_{n+i} = 0 - 1\pi_i.
$$

Combining this expression with (10), we obtain:

$$
\bar{c}_{n+i} = -\pi_i = -y_i.
$$

Thus the shadow prices y_i are the multiples π_i.

Since these multiples can be used to obtain every objective coefficient in the final form (9), the reduced cost $\bar{c}_j$ of variable x_j is given by:

$$\bar{c}_j = c_j - \sum_{i=1}^{m} a_{ij} y_i \qquad (j = 1, 2, \ldots, n), \tag{11}$$

and the current value of the objective function is:

$$-\bar{z}_0 = -\sum_{i=1}^{m} b_i y_i$$

or, equivalently,

$$\bar{z}_0 = \sum_{i=1}^{m} b_i y_i \tag{12}$$

Expression (11) links the shadow prices to the reduced cost of each variable, while (12) establishes the relationship between the shadow prices and the optimal value of the objectivefunction.

Expression (11) also can be viewed as a mathematical definition of the shadow prices. Since $\bar{c}_j = 0$ for the m basic variables of the optimal solution, we have:

$$0 = c_j - \sum_{i=1}^{m} a_{ij} y_i \qquad \text{for } j \text{ basic.}$$

This is a system of m equations in m unknowns that uniquely determines the values of the shadow prices y_i.

3.3 VARIATIONS IN THE OBJECTIVE COEFFICIENTS

Now let us consider the question of how much the objective-function coefficients can vary without changing the values of the decision variables in the optimal solution. We will make the changes one at a time, holding all other coefficients and righthand-side values constant. The reason for this is twofold: first, the calculation of the range on one coefficient is fairly simple and therefore not expensive; and second, describing ranges when more than two coefficients are simultaneously varied would require a system of equations instead of a simple interval.

We return to consideration of the objective coefficient of x_3, a nonbasic variable in our example. Clearly, if the contribution is reduced from $6 per case to something less it would certainly not become attractive to produce champagne glasses. If it is now not attractive to produce champagne glasses, then *reducing* the contribution from their production only makes it less attractive. However, if the contribution from production of champagne glasses is increased, presumably there is some level of contribution such that it becomes attractive to produce them. In fact, it was argued above that the opportunity cost associated with diverting resources from the optimal production schedule was merely the shadow price associated with a resource multiplied by the amount of the resource consumed by the activity. For this activity, the opportunity cost is $6\frac{4}{7}$ per case compared to a contribution of $6 per case. If the contribution were increased above the break-even opportunity cost, then it would become attractive to produce champagne glasses.

Let us relate this to the procedures of the simplex method. Suppose that we increase the objective function coefficient of x_3 in the original problem formulation (5) by Δc_3, giving us:

$$5x_1 + 4.5x_2 + \underbrace{(6 + \Delta c_3)}_{= \, c_3^{\text{new}}} x_3 - z = 0.$$

Tableau 3

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_2	$4\frac{2}{7}$		1	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{3}{35}$	
x_6	$1\frac{4}{7}$			$-\frac{11}{7}$	$-\frac{2}{7}$	$\frac{1}{14}$	1
x_1	$6\frac{3}{7}$	1		$\frac{11}{7}$	$\frac{2}{7}$	$-\frac{1}{14}$	
$(-z)$	$-51\frac{3}{7}$			$-\frac{4}{7} + \Delta c_3$	$-\frac{11}{14}$	$-\frac{1}{35}$	

In applying the simplex method, multiples of the rows were subtracted from the objective function to yield the final system of equations. Therefore, the objective function in the final tableau will remain unchanged except for the addition of $\Delta c_3 x_3$. The modified final tableau is given in Tableau 3 .

Now x_3 will become a candidate to enter the optimal solution at a positive level, i.e., to enter the basis, only when its objective-function coefficient is positive. The optimal solution remains unchanged so long as:

$$-\tfrac{4}{7} + \Delta c_3 \le 0 \qquad \text{or} \qquad \Delta c_3 \le \tfrac{4}{7}.$$

Equivalently, we know that the range on the original objective-function coefficient of x_3, say c_3^{new}, must satisfy

$$-\infty < c_3^{\text{new}} \le 6\tfrac{4}{7}$$

if the optimal solution is to remain unchanged.

Next, let us consider what happens when the objective-function coefficient of a basic variable is varied. Consider the range of the objective-function coefficient of variable x_1. It should be obvious that if the contribution associated with the production of six-ounce juice glasses is reduced sufficiently, we will stop producing them. Also, a little thought tells us that if the contribution were *increased* sufficiently, we might end up producing *only* six-ounce juice glasses. To understand this mathematically, let us start out as we did before by adding Δc_1 to the objective-function coefficient of x_1 in the original problem formulation (5) to yield the following modified objective function:

$$(5 + \Delta c_1)x_1 + 4.5x_2 + 6x_3 - z = 0.$$

If we apply the same logic as in the case of the nonbasic variable, the result is Tableau 4.

Tableau 4

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_2	$4\frac{2}{7}$		1	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{3}{35}$	
x_6	$1\frac{4}{7}$			$-\frac{11}{7}$	$-\frac{2}{7}$	$\frac{1}{14}$	1
x_1	$6\frac{3}{7}$	1		$\frac{11}{7}$	$\frac{2}{7}$	$-\frac{1}{14}$	
$(-z)$	$-51\frac{3}{7}$	Δc_1		$-\frac{4}{7}$	$-\frac{11}{14}$	$-\frac{1}{35}$	

However, the simplex method requires that the final system of equations be in canonical form with respect to the basic variables. Since the basis is to be unchanged, in order to make the coefficient of x_1 zero in the final tableau we must subtract Δc_1 times row 3 from row 4 in Tableau 4. The result is Tableau 5.

By the simplex optimality criterion, all the objective-function coefficients in the final tableau must be

Tableau 5

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_2	$4\frac{2}{7}$		1	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{3}{35}$	
x_6	$1\frac{4}{7}$			$-\frac{11}{7}$	$-\frac{2}{7}$	$\frac{1}{14}$	1
x_1	$6\frac{3}{7}$	1		$\frac{11}{7}$	$\frac{2}{7}$	$-\frac{1}{14}$	
$(-z)$	$-51\frac{3}{7} - 6\frac{3}{7}\Delta c_1$			$-\frac{4}{7} - \frac{11}{7}\Delta c_1$	$-\frac{11}{14} - \frac{2}{7}\Delta c_1$	$-\frac{1}{35} + \frac{1}{14}\Delta c_1$	

nonpositive in order to have the current solution remain unchanged. Hence, we must have:

$$-\tfrac{4}{7} - \tfrac{11}{7}\Delta c_1 \le 0 \qquad \left(\text{that is, } \Delta c_1 \ge -\tfrac{4}{11}\right),$$

$$-\tfrac{11}{14} - \tfrac{2}{7}\Delta c_1 \le 0 \qquad \left(\text{that is, } \Delta c_1 \ge -\tfrac{11}{4}\right),$$

$$-\tfrac{1}{35} + \tfrac{1}{14}\Delta c_1 \le 0 \qquad \left(\text{that is, } \Delta c_1 \le +\tfrac{2}{5}\right);$$

and, taking the most limiting inequalities, the bounds on Δc_1 are:

$$-\tfrac{4}{11} \le \Delta c_1 \le \tfrac{2}{5}.$$

If we let $c_1^{\text{new}} = c_1 + \Delta c_1$ be the objective-function coefficient of x_1 in the initial tableau, then:

$$4\tfrac{7}{11} \le c_1^{\text{new}} \le 5\tfrac{2}{5},$$

where the current value of $c_1 = 5$.

   It is easy to determine which variables will enter and leave the basis when the new cost coefficient reaches either of the extreme values of the range. When $c_1^{\text{new}} = 5\frac{2}{5}$, the objective coefficient of x_5 in the final tableau becomes 0; thus x_5 enters the basis for any further increase of c_1^{new}. By the usual ratio test of the simplex method,

$$\underset{i}{\text{Min}}\left\{\frac{\bar{b}_i}{\bar{a}_{is}}\,\middle|\, \bar{a}_{is} > 0\right\} = \text{Min}\left\{\frac{4\frac{2}{7}}{\frac{3}{35}}, \frac{1\frac{4}{7}}{\frac{1}{14}}\right\} = \text{Min }\{50, 22\},$$

and the variable x_6, which is basic in row 2, leaves the basis when x_5 is introduced. Similarly, when $c_1^{\text{new}} = 4\frac{7}{11}$, the objective coefficient of x_3 in the final tableau becomes 0, and x_3 is the entering variable. In this case, the ratio test shows that x_1 leaves the basis.

General Discussion

To determine the ranges of the cost coefficients in the optimal solution of any linear program, it is useful to distinguish between nonbasic variables and basic variables.

   If x_j is a nonbasic variable and we let its objective-function coefficient c_j be changed by an amount Δc_j with all other data held fixed, then the current solution remains unchanged so long as the new reduced cost $\bar{c}_j^{\text{new}}$ remains nonnegative, that is,

$$\bar{c}_j^{\text{new}} = c_j + \Delta c_j - \sum_{i=1}^{m} a_{ij} y_i = \bar{c}_j + \Delta c_j \le 0.$$

The range on the variation of the objective-function coefficient of a nonbasic variable is then given by:

$$-\infty < \Delta c_j \le -\bar{c}_j, \tag{13}$$

so that the range on the objective-function coefficient $c_j^{\text{new}} = c_j + \Delta c_j$ is:

$$-\infty < c_j^{\text{new}} \leq c_j - \bar{c}_j.$$

If x_r is a basic variable in row k and we let its original objective-function coefficient c_r be changed by an amount Δc_r with all other data held fixed, then the coefficient of the variable x_r in the final tableau changes to:

$$\bar{c}_r^{\text{new}} = c_r + \Delta c_r - \sum_{i=1}^{m} a_{ir} y_i = \bar{c}_r + \Delta c_r.$$

Since x_r is a basic variable, $\bar{c}_r = 0$; so, to recover a canonical form with $\bar{c}_r^{\text{new}} = 0$, we subtract Δc_r times the kth constraint in the final tableau from the final form of the objective function, to give new reduced costs for all nonbasic variables,

$$\bar{c}_j^{\text{new}} = \bar{c}_j - \Delta c_r \bar{a}_{kj}. \tag{14}$$

Here $\bar{a}_{kj}$ is the coefficient of variable x_j in the kth constraint of the final tableau. Note that $\bar{c}_j^{\text{new}}$ will be zero for all basic variables.

The current basis remains optimal if $\bar{c}_j^{\text{new}} \leq 0$. Using this condition and (14), we obtain the range on the variation of the objective-function coefficient:

$$\text{Max}_{j} \left\{ \frac{\bar{c}_j}{\bar{a}_{kj}} \,\middle|\, \bar{a}_{kj} > 0 \right\} \leq \Delta c_r \leq \text{Min}_{j} \left\{ \frac{\bar{c}_j}{\bar{a}_{kj}} \,\middle|\, \bar{a}_{kj} < 0 \right\}. \tag{15}$$

The range on the objective-function coefficient $c_r^{\text{new}} = c_r + \Delta c_r$ of the basic variable x_r is determined by adding c_r to each bound in (15).

The variable transitions that occur at the limits of the cost ranges are easy to determine. For nonbasic variables, the entering variable is the one whose cost is being varied. For basic variables, the entering variable is the one giving the limiting value in (15). The variable that leaves the basis is then determined by the minimum-ratio rule of the simplex method. If x_s is the entering variable, then the basic variable in row r, determined by:

$$\frac{\bar{b}_r}{\bar{a}_{rs}} = \text{Min}_{i} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \,\middle|\, \bar{a}_{is} > 0 \right\},$$

is dropped from the basis.

Since the calculation of these ranges and the determination of the variables that will enter and leave the basis if a range is exceeded are computationally inexpensive to perform, this information is invariably reported on any commercially available computer package. These computations are easy since no iterations of the simplex method need be performed. It is necessary only (1) to check the entering variable conditions of the simplex method to determine the ranges, as well as the variable that enters the basis, and (2) to check the leaving variable condition (i.e., the minimum-ratio rule) of the simplex method to compute the variable that leaves the basis.

3.4 VARIATIONS IN THE RIGHTHAND-SIDE VALUES

Now let us turn to the questions related to the righthand-side ranges. We already have noted that a righthand-side range is the interval over which an individual righthand-side value can be varied, all the other problem data being held constant, such that variables that constitute the basis remain the same. Over these ranges, the *values* of the decision variables are clearly modified. Of what use are these righthand-side ranges? Any change in the righthand-side values that keep the current basis, and therefore the canonical form, unchanged has no effect upon the objective-function coefficients. Consequently, the righthand-side ranges are such that the *shadow prices* (which are the negative of the coefficients of the slack or artificial variables in the final tableau) and the *reduced costs* remain unchanged for variations of a single value within the stated range.

We first consider the righthand-side value for the demand limit on six-ounce juice glasses in our example. Since this constraint is not binding, the shadow price associated with it is zero and it is simple to determine the appropriate range. If we add an amount Δb_3 to the righthand side of this constraint (4), the constraint changes to:

$$x_1 + x_6 = 8 + \Delta b_3.$$

In the original problem formulation, it should be clear that, since x_6, the slack in this constraint, is a basic variable in the final system of equations, x_6 is merely increased or decreased by Δb_3. In order to keep the current solution feasible, x_6 must remain greater than or equal to zero. From the final tableau, we see that the current value of $x_6 = 1\frac{4}{7}$; therefore x_6 remains in the basis if the following condition is satisfied:

$$x_6 = 1\frac{4}{7} + \Delta b_3 \geq 0.$$

This implies that:

$$\Delta b_3 \geq -1\frac{4}{7}$$

or, equivalently, that:

$$b_3^{\text{new}} = 8 + \Delta b_3 \geq 6\frac{3}{7}.$$

Now let us consider changing the righthand-side value associated with the storage-capacity constraint of our example. If we add Δb_2 to the righthand-side value of constraint (3), this constraint changes to:

$$10x_1 + 20x_2 + 10x_3 + x_5 = 150 + \Delta b_2.$$

In the original problem formulation, as was previously remarked, changing the righthand-side value is essentially equivalent to decreasing the value of the slack variable x_5 of the corresponding constraint by Δb_2; that is, substituting $x_5 - \Delta b_2$ for x_5 in the original problem formulation. In this case, x_5, which is zero in the final solution, is changed to $x_5 = -\Delta b_2$. We can analyze the implications of this increase in the righthand-side value by using the relationships among the variables represented by the final tableau. Since we are allowing only one value to change at a time, we will maintain the remaining nonbasic variables, x_3 and x_4, at zero level, and we let $x_5 = -\Delta b_2$. Making these substitutions in the final tableau provides the following relationships:

$$x_2 - \tfrac{3}{35}\Delta b_2 = 4\tfrac{2}{7},$$
$$x_6 - \tfrac{1}{14}\Delta b_2 = 1\tfrac{4}{7},$$
$$x_1 + \tfrac{1}{14}\Delta b_2 = 6\tfrac{3}{7}.$$

In order for the current basis to remain optimal, it need only remain feasible, since the reduced costs will be unchanged by any such variation in the righthand-side value. Thus,

$$x_2 = 4\tfrac{2}{7} + \tfrac{3}{35}\Delta b_2 \geq 0 \qquad \text{(that is, } \Delta b_2 \geq -50\text{)},$$
$$x_6 = 1\tfrac{4}{7} + \tfrac{1}{14}\Delta b_2 \geq 0 \qquad \text{(that is, } \Delta b_2 \geq -22\text{)},$$
$$x_1 = 6\tfrac{3}{7} - \tfrac{1}{14}\Delta b_2 \geq 0 \qquad \text{(that is, } \Delta b_2 \leq 90\text{)},$$

which implies:

$$-22 \leq \Delta b_2 \leq 90,$$

or, equivalently,

$$128 \leq b_2^{\text{new}} \leq 240,$$

where the current value of $b_2 = 150$ and $b_2^{\text{new}} = 150 + \Delta b_2$.

Observe that these computations can be carried out directly in terms of the final tableau. When changing the ith righthand side by Δb_i, we simply substitute $-\Delta b_i$ for the slack variable in the corresponding constraint and update the current values of the basic variables accordingly. For instance, the change of storage capacity just considered is accomplished by setting $x_5 = -\Delta b_2$ in the final tableau to produce Tableau 6 with modified current values.

Note that the change in the optimal value of the objective function is merely the shadow price on the storage-capacity constraint multiplied by the increased number of units of storage capacity.

Tableau 6

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_2	$4\frac{2}{7} + \frac{3}{35}\Delta b_2$		1	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{3}{35}$	
x_6	$1\frac{4}{7} + \frac{1}{14}\Delta b_2$			$-\frac{11}{7}$	$-\frac{2}{7}$	$\frac{1}{14}$	1
x_1	$6\frac{3}{7} - \frac{1}{14}\Delta b_2$	1		$\frac{11}{7}$	$\frac{2}{7}$	$-\frac{1}{14}$	
$(-z)$	$-51\frac{3}{7} - \frac{1}{35}\Delta b_2$			$-\frac{4}{7}$	$-\frac{11}{14}$	$-\frac{1}{35}$	

Variable Transitions

We have just seen how to compute righthand-side ranges so that the current basis remains optimal. For example, when changing demand on six-ounce juice glasses, we found that the basis remains optimal if

$$b_3^{\text{new}} \geq 6\tfrac{3}{7},$$

and that for $b_3^{\text{new}} < 6\frac{3}{7}$, the basic variable x_6 becomes negative. If b_3^{new} were reduced below $6\frac{3}{7}$, what change would take place in the basis? First, x_6 would have to leave the basis, since otherwise it would become negative. What variable would then enter

the basis to take its place? In order to have the new basis be an optimal solution, the entering variable must be chosen so that the reduced costs are not allowed to become positive.

Regardless of which variable enters the basis, the entering variable will be isolated in row 2 of the final tableau to replace x_6, which leaves the basis. To isolate the entering variable, we must perform a pivot operation, and a multiple, say t, of row 2 in the final tableau will be subtracted from the objective-function row. Assuming that b_3^{new} were set equal to $6\frac{3}{7}$ in the initial tableau, the results are given in Tableau 7.

Tableau 7

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_2	$4\frac{2}{7}$		1	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{3}{35}$	
x_6	0			$-\frac{11}{7}$	$-\frac{2}{7}$	$\frac{1}{14}$	1
x_1	$6\frac{3}{7}$	1		$\frac{11}{7}$	$\frac{2}{7}$	$-\frac{1}{14}$	
$(-z)$	$-51\frac{3}{7}$			$-\frac{4}{7} + \frac{11}{7}t$	$-\frac{11}{14} + \frac{2}{7}t$	$-\frac{1}{35} - \frac{1}{14}t$	$-t$

In order that the new solution be an optimal solution, the coefficients of the variables in the objective function of the final tableau must be nonpositive; hence,

$$-\tfrac{4}{7} + \tfrac{11}{7}t \leq 0 \quad \left(\text{that is, } t \leq \tfrac{4}{11}\right),$$

$$-\tfrac{11}{14} + \tfrac{2}{7}t \leq 0 \quad \left(\text{that is, } t \leq \tfrac{11}{4}\right),$$

$$-\tfrac{1}{35} - \tfrac{1}{14}t \leq 0 \quad \left(\text{that is, } t \geq -\tfrac{2}{5}\right),$$

$$-t \leq 0 \quad (\text{that is, } t \geq 0),$$

which implies:

$$0 \leq t \leq \tfrac{4}{11}.$$

Since the coefficient of x_3 is most constraining on t, x_3 will enter the basis. Note that the range on the righthand-side value and the variable transitions that would occur if that range were exceeded by a small amount are

easily computed. However, the pivot operation actually introducing x_3 into the basis and eliminating x_6 need not be performed.

As another example, when the change Δb_2 in storage capacity reaches -22 in Tableau 6, then x_6, the slack on six-ounce juice glasses, reaches zero and will drop from the basis, and again x_3 enters the basis. When $\Delta b_2 = 90$, though, then x_1, the production of six-ounce glasses, reaches zero and will drop from the basis. Since x_1 is a basic variable in the third constraint of Tableau 6, we must pivot in the third row, subtracting t times this row from the objective function. The result is given in Tableau 8.

Tableau 8

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_2	12		1	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{3}{35}$	
x_6	8			$-\frac{11}{7}$	$-\frac{2}{7}$	$\frac{1}{14}$	1
x_1	0	1		$\frac{11}{7}$	$\frac{2}{7}$	$-\frac{1}{14}$	
$(-z)$	-54	$-t$		$-\frac{4}{7}-\frac{11}{7}t$	$-\frac{11}{7}-\frac{2}{7}t$	$-\frac{1}{35}+\frac{1}{14}t$	

The new objective-function coefficients must be nonpositive in order for the new basis to be optimal; hence,

$$-t \le 0 \qquad \text{(that is, } t \ge 0\text{)},$$

$$-\tfrac{4}{7}-\tfrac{11}{7}t \le 0 \qquad \left(\text{that is, } t \ge -\tfrac{4}{11}\right),$$

$$-\tfrac{11}{7}-\tfrac{2}{7}t \le 0 \qquad \left(\text{that is, } t \ge -\tfrac{11}{2}\right),$$

$$-\tfrac{1}{35}+\tfrac{1}{14}t \le 0 \qquad \left(\text{that is, } t \le \tfrac{2}{5}\right),$$

which implies $0 \le t \le \frac{2}{5}$. Consequently, x_5 enters the basis. The implication is that, if the storage capacity were increased from 150 to 240 cubic feet, then we would produce only the ten-ounce cocktail glasses, which have the highest contribution per hour of production time.

General Discussion

In the process of solving linear programs by the simplex method, the initial canonical form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} &= b_1,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad\quad + x_{n+2} &= b_2,\\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \ddots \qquad &\quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \underbrace{\qquad\qquad\qquad + x_{n+m}}_{\text{Original basic variables}} &= b_m
\end{aligned}
$$

is transformed into a canonical from:

$$
\begin{aligned}
\bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n + \beta_{11}x_{n+1} + \beta_{12}x_{n+2} + \cdots + \beta_{1m}x_{n+m} &= \bar{b}_1,\\
\bar{a}_{21}x_1 + \bar{a}_{22}x_2 + \cdots + \bar{a}_{2n}x_n + \beta_{21}x_{n+1} + \beta_{22}x_{n+2} + \cdots + \beta_{2m}x_{n+m} &= \bar{b}_2,\\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad &\quad \vdots\\
\bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \cdots + \bar{a}_{mn}x_n + \beta_{m1}x_{n+1} + \beta_{m2}x_{n+2} + \cdots + \beta_{mn}x_{n+m} &= \bar{b}_m.
\end{aligned}
$$

Of course, because this is a canonical form, the updated data $\bar{a}_{ij}$ and β_{ij} will be structured so that one basic variable is isolated in each constraint. Since the updated coefficients of the initial basic (slack or artificial)

variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ in the final tableau play such an important role in sensitivity analysis, we specifically denote these values as β_{ij}.

We can change the coefficient b_k of the kth righthand side in the initial tableau by Δb_k with all the other data held fixed, simply by substituting $x_{n+k} - \Delta b_k$ for x_{n+k} in the original tableau. To see how this change affects the updated righthand-side coefficients, we make the same substitution in the final tableau. Only the terms $\beta_{ik} x_{n+k}$ for $i = 1, 2, \ldots, m$ change in the final tableau. They become $\beta_{ik}(x_{n+k} - \Delta b_k) = \beta_{ik} x_{n+k} - \beta_{ik} \Delta b_k$. Since $\beta_{ik} \Delta b_k$ is a constant, we move it to the righthand side to give modified righthand-side values:

$$\bar{b}_i + \beta_{ik} \Delta b_k \qquad (i = 1, 2, \ldots, m).$$

As long as all of these values are nonnegative, the basis specified by the final tableau remains optimal, since the reduced costs have not been changed. Consequently, the current basis is optimal whenever $\bar{b}_i + \beta_{ik} \Delta b_k \geq 0$ for $i = 1, 2, \ldots, m$ or, equivalently,

$$\underset{i}{\text{Max}} \left\{ \frac{-\bar{b}_i}{\beta_{ik}} \middle| \beta_{ik} > 0 \right\} \leq \Delta b_k \leq \underset{i}{\text{Min}} \left\{ \frac{-b_i}{\beta_{ik}} \middle| \beta_{ik} < 0 \right\}. \tag{16}$$

The lower bound disappears if all $\beta_{ik} \leq 0$, and the upper bound disappears if all $\beta_{ik} \geq 0$.

When Δb_k reaches either its upper or lower bound in Eq. (16), any further increase (or decrease) in its value makes one of the updated righthand sides, say $\bar{b}_r + \beta_{rk} \Delta b_k$, negative. At this point, the basic variable in row r leaves the basis, and we must pivot in row r in the final tableau to find the variable to be introduced in its place. Since pivoting subtracts a multiple t of the pivot row from the objective equation, the new objective equation has coefficients:

$$\bar{c}_j - t \bar{a}_{rj} \qquad (j = 1, 2, \ldots, n). \tag{17}$$

For the new basis to be optimal, each of these coefficients must be nonpositive. Since $\bar{c}_j = 0$ for the basic variable being dropped and its coefficient in constraint r is $\bar{a}_{rj} = 1$, we must have $t \geq 0$. For any nonnegative t, the updated coefficient $\bar{c}_j - t \bar{a}_{rj}$ for any other variable remains nonpositive if $\bar{a}_{rj} \geq 0$. Consequently we need only consider $\bar{a}_{rj} < 0$, and t is given by

$$t = \underset{j}{\text{Min}} \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} \middle| \bar{a}_{rj} < 0 \right\}. \tag{18}$$

The index s giving this minimum has $\bar{c}_s - t \bar{a}_{rs} = 0$, and the corresponding variable x_s can become the new basic variable in row r by pivoting on $\bar{a}_{rs}$. Note that this pivot is made on a negative coefficient.

Since the calculation of the righthand-side ranges and the determination of the variables that will enter and leave the basis when a range is exceeded are computationally easy to perform, this information is reported by commercially available computer packages. For righthand-side ranges, it is necessary to check only the feasibility conditions to determine the ranges as well as the variable that leaves, and it is necessary to check only the entering variable condition (18) to complete the variable transitions. This condition will be used again in Chapter 4, since it is the minimum-ratio rule of the so-called dual simplex method.

3.5 ALTERNATIVE OPTIMAL SOLUTIONS AND SHADOW PRICES

In many applied problems it is important to identify alternative optimal solutions when they exist. When there is more than one optimal solution, there are often good external reasons for preferring one to the other; therefore it is useful to be able to easily determine alternative optimal solutions.

As in the case of the objective function and righthand-side ranges, the final tableau of the linear program tells us something conservative about the possibility of alternative optimal solutions. First, if all reduced costs of the nonbasic variables are strictly negative (positive) in a maximization (minimization) problem, then there is *no* alternative optimal solution, because introducing any variable into the basis at a positive level would reduce (increase) the value of the objective function. On the other hand, if one or more of the reduced costs are zero, there *may* exist alternative optimal solutions. Suppose that, in our custom-molder example,

the contribution from champagne glasses, x_3, had been $6\frac{4}{7}$. From Section 3.1 we know that the reduced cost associated with this activity would be zero. The final tableau would look like Tableau 9.

Tableau 9

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	Ratio test
x_2	$4\frac{2}{7}$		1	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{3}{35}$		
x_6	$1\frac{4}{7}$			$-\frac{11}{7}$	$-\frac{2}{7}$	$\frac{1}{14}$	1	
x_1	$6\frac{3}{7}$	1		$\frac{11}{7}$	$\frac{2}{7}$	$-\frac{1}{14}$		$6\frac{3}{7}/\frac{11}{7}$
$(-z)$	$-51\frac{3}{7}$			0	$-\frac{11}{14}$	$-\frac{1}{35}$		

$\uparrow$

This would imply that x_3, production of champagne glasses, could be introduced into the basis without changing the value of the objective function. The variable that would be dropped from the basis would be determined by the usual minimum-ratio rule:

$$\operatorname*{Min}_{i} \left\{ \frac{\bar{b}_i}{\bar{a}_{i3}} \,\middle|\, \bar{a}_{i3} > 0 \right\} = \operatorname{Min} \left\{ \frac{4\frac{2}{7}}{\frac{2}{7}}, \frac{6\frac{3}{7}}{\frac{11}{7}} \right\} = \operatorname{Min} \left\{ 14, \frac{45}{11} \right\}.$$

In this case, the minimum ratio occurs for row 3 so that x_1 leaves the basis. The alternative optimal solution is then found by completing the pivot that introduces x_3 and eliminates x_1. The values of the new basic variables can be determined from the final tableau as follows:

$$x_3 = \tfrac{45}{11},$$
$$x_2 = 4\tfrac{2}{7} - \tfrac{2}{7}x_3 = 4\tfrac{2}{7} - \tfrac{2}{7}\left(\tfrac{45}{11}\right) = 3\tfrac{9}{77},$$
$$x_6 = 1\tfrac{4}{7} + \tfrac{11}{7}x_3 = 1\tfrac{4}{7} + \tfrac{11}{7}\left(\tfrac{45}{11}\right) = 8,$$
$$x_1 = 6\tfrac{3}{7} - \tfrac{11}{7}x_3 = 6\tfrac{3}{7} - \tfrac{11}{7}\left(\tfrac{45}{11}\right) = 0,$$
$$z = 51\tfrac{3}{7}.$$

Under the assumption that the contribution from champagne-glass production is $6\frac{4}{7}$, we have found an alternative optimal solution. It should be pointed out that any weighted combination of these two solutions is then also an alternative optimal solution.

In general, we can say that there *may* exist an alternative optimal solution to a linear program if one or more of the reduced costs of the nonbasic variables are zero. There *does* exist an alternative optimal solution if one of the nonbasic variables with zero reduced cost can be introduced into the basis at a positive level. In this case, any weighted combination of these solutions is also an alternative optimal solution. However, if it is not possible to introduce a new variable at a positive level, then no such alternative optimal solution exists even though some of the nonbasic variables have zero reduced costs. Further, the problem of finding *all* alternative optimal solutions cannot be solved by simply considering the reduced costs of the final tableau, since there can in general exist alternative optimal solutions that cannot be reached by a *single* pivot operation.

Independent of the question of whether or not alternative optimal solutions exist in the sense that different values of the decision variables yield the same optimal value of the objective function, there may exist alternative optimal shadow prices. The problem is completely analogous to that of identifying alternative optimal solutions. First, if all righthand-side values in the final tableau are positive, then there do not exist alternative optimal shadow prices. Alternatively, if one or more of these values are zero, then there *may* exist

alternative optimal shadow prices. Suppose, in our custom-molder example, that the value of storage capacity had been 128 hundred cubic feet rather than 150. Then from Tableau 6 in Section 3.4, setting $\Delta b_2 = -22$, we illustrate this situation in Tableau 10.

Tableau 10

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_2	$\frac{12}{5}$		1	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{3}{35}$	
x_6	0			$-\frac{11}{7}$	$-\frac{2}{7}$	$\frac{1}{14}$	1
x_1	8	1		$\frac{11}{7}$	$\frac{2}{7}$	$-\frac{1}{14}$	
$(-z)$	$-50\frac{4}{5}$			$-\frac{4}{7}$	$-\frac{11}{14}$	$-\frac{2}{70}$	

Since the righthand-side value in row 2 of Tableau 10 is zero, it is possible to drop x_6, the basic variable in row 2 of the final tableau, from the basis as long as there is a variable to introduce into the basis. The variable to be introduced into the basis can be determined from the entering variable condition, Eq. (18):

$$\text{Min}_j \left\{ \frac{\bar{c}_j}{\bar{a}_{2j}} \,\middle|\, \bar{a}_{2j} < 0 \right\} = \text{Min} \left\{ \frac{-\frac{4}{7}}{-\frac{11}{7}}, \frac{-\frac{11}{14}}{-\frac{2}{7}} \right\} = \text{Min} \left\{ \frac{4}{11}, \frac{11}{4} \right\}.$$

In this case the minimum ratio implies that the production of champagne glasses, x_3, enters the basis. The values of the reduced costs for the new basis can be determined by subtracting $\frac{4}{11}$ times row 2 in the final tableau from the objective function in the final tableau. Thus we have:

$$\bar{c}_3 = -\frac{4}{7} + \frac{11}{7}\left(\frac{4}{11}\right) = 0,$$

$$\bar{c}_4 = -\frac{11}{14} + \frac{2}{7}\left(\frac{4}{11}\right) = -\frac{15}{22},$$

$$\bar{c}_5 = -\frac{1}{35} - \frac{1}{14}\left(\frac{4}{11}\right) = -\frac{3}{55},$$

$$\bar{c}_6 = -\frac{4}{11}.$$

Since the shadow prices are the negative of the objective-function coefficients of the slack variables in the final tableau, the alternative optimal shadow prices in this case are $\frac{15}{22}$, $\frac{3}{55}$, and $\frac{4}{11}$ for the three constraints, respectively.

In general we can say that there *may* exist alternative optimal shadow prices for a linear program if one or more of the righthand-side values in the final tableau are zero. There *does* exist an alternative set of optimal shadow prices if the variable to enter the basis determined by the minimum-ratio rule has a strictly negative reduced cost. As in the case of alternative optimal solutions, any weighted combination of these sets of shadow prices is also an alternative set of optimal shadow prices. Finding all such sets of alternative optimal shadow prices is of the same degree of complexity as finding all alternative optimal solutions, since in general there can exist alternative sets of shadow prices that cannot be reached in a *single* pivot. Finally, it should be pointed out that all four cases can take place: unique solution with unique shadow prices; unique solution with alternative optimal shadow prices; alternative optimal solutions with unique shadow prices; and alternative optimal solutions and alternative shadow prices simultaneously.

3.6 THE COMPUTER OUTPUT—AN EXAMPLE

We have remarked that any commercially available linear-programming package routinely reports a great deal of information besides the values of the decision variables at the optimal solution. In this section we

illustrate the typical computer output for the variation of the custom-molder example presented in the first few sections of this chapter. The results shown in Fig. 3.1 were obtained on a commercial time-sharing system.* Note that the output includes a tabular display of the problem as stated in Eq. (1).

The first four numbered sections of this output should be compared to Tableau 12 for this example, given in Section 3.1. The optimal value of the objective function is $z = 51\frac{3}{7}$ hundred dollars, while the associated values of the decision variables are as follows: production of six-ounce juice glasses $x_1 = 6\frac{3}{7}$ hundred cases, and production of ten-ounce cocktail glasses $x_2 = 4\frac{2}{7}$ hundred cases. Note that there is slack in the constraint on demand for six-ounce juice glasses of $1\frac{4}{7}$ hundred cases, which corresponds to variable x_6. Finally, champagne glasses are not produced, and hence $x_3 = 0$. Note that the reduced cost associated with production of champagne glasses is $-\frac{4}{7}$, which is the amount the objective function would decrease per hundred cases if champagne glasses were in fact produced.

In our discussion of shadow prices in Section 3.2, it is pointed out that the shadow prices are the negative of the reduced costs associated with the slack variables. Thus the shadow price on production capacity is $\frac{11}{14}$ hundred dollars per hour of production time, and the shadow price on storage capacity is $\frac{1}{35}$ hundred dollars per hundred square feet of storage space. The shadow price on six-ounce juice glasses demand is zero, since there remains unfulfilled demand in the optimal solution. It is intuitive that, in general, either the shadow price associated with a constraint is nonzero or the slack (surplus) in that constraint is nonzero, but both will not simultaneously be nonzero. This property is referred to as *complementary slackness*, and is discussed in Chapter 4.

Sections ∗5∗ and ∗7∗ of the computer output give the ranges on the coefficients of the objective function and the variable transitions that take place. (This material is discussed in Section 3.3.) Note that the range on the nonbasic variable x_3, production of champagne glasses, is one-sided. Champagne glasses are not currently produced when their contribution is $6 per case, so that, if their contribution were reduced, we would certainly not expect this to change. However, if the contribution from the production of champagne glasses is increased to $6\frac{4}{7}$ per case, their production becomes attractive, so that variable x_3 would enter the basis and variable x_1, production of six-ounce juice glasses, would leave the basis. Consider now the range on the coefficient of x_1, production of six-ounce juice glasses, where the current contribution is $5 per case. If this contribution were raised to $5\frac{2}{5}$ per case, the slack in storage capacity would enter the basis, and the slack in juice glass, demand would leave. This means we would meet all of the juice-glass demand and storage capacity would no longer be binding. On the other hand, if this contribution were reduced to $4\frac{4}{7}$ per case, variable x_3, production of champagne glasses, would enter the basis and variable x_1, production of six-ounce juice-glasses, would leave. In this instance, juice glasses would no longer be produced.

Sections ∗6∗ and ∗8∗ of the computer output give the ranges on the righthand-side values and the variable transitions that result. (This material is discussed in Section 3.4.) Consider, for example, the range on the righthand-side value of the constraint on storage capacity, where the current storage capacity is 150 hundred square feet. If this value were increased to 240 hundred square feet, the slack in storage capacity would enter the basis and variable x_1, production of six-ounce glasses, would leave. This means we would no longer produce six-ounce juice glasses, but would devote all our production capacity to ten-ounce cocktail glasses. If the storage capacity were reduced to 128 hundred square feet, we would begin producing champagne glasses and the slack in the demand for six-ounce juice glasses would leave the basis. The other ranges have similar interpretations.

Thus far we have covered information that is available routinely when a linear program is solved. The format of the information varies from one computer system to another, but the information available is always the same. The role of computers in solving linear-programming problems is covered more extensively in Chapter 5.

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* Excel spreadsheets available at `http://web.mit.edu/15.053/www/Sect3.6_Glasses.xls` and `http://web.mit.edu/15.053/www/Sect3.6_Sensitivity.xls`

```
TITLE: CUSTOM MOLDER
PROCEED, DISPLAY, OR REJECT? DISPLAY

FORMAT? 0

OBJECTIVES:

              SIX-OZ    TEN-OZ    CHAMP

CONTRIB      5.000     4.500     6.000


CONSTRAINTS:

              SIX-OZ    TEN-OZ    CHAMP     RELATION  RHS

PROD-HR      6.000     5.000     8.000     LE        60.00
STORAGE      10.00     20.00     10.00     LE        150.0
DEMAND       1.000     .0000     .0000     LE        8.000

PROCEED OR REJECT? PROCEED

PARAMETRICS? NO

MAXIMIZE OR MINIMIZE? MAX

OPTIMAL SOLUTION FOUND.
     CONTRIB      51.4286

OUTPUT OPTION? ALL

ALL ITEMS NOT LISTED IN SECTIONS 1 - 4 HAVE THE VALUE ZERO.

*1* DECISION VARIABLES
   1. SIX-OZ    6.42857
   2. TEN-OZ    4.28571

*2* SLACK(+) AND SURPLUS(-) IN CONSTRAINTS
   3. +DEMAND   1.57143

*3* SHADOW PRICES FOR CONSTRAINTS
   1. PROD-HR   .785714
   2. STORAGE   .285714E-01

*4* REDUCED COSTS FOR DECISION VARIABLES
   3. CHAMP     -.571429

*5* RANGES ON COEFFICIENTS OF OBJECTIVE CONTRIB
        VARIABLE  LOWER BOUND  CURRENT VALUE  UPPER BOUND
   1. SIX-OZ      4.6364        5.0000         5.4000
   2. TEN-OZ      4.1667        4.5000         6.5000
   3. CHAMP       UNBOUNDED     6.0000         6.5714

*6* RANGES ON VALUES OF RIGHT-HAND-SIDE RHS
        CONSTRNT  LOWER BOUND  CURRENT VALUE  UPPER BOUND
   1.  PROD-HR    37.500        60.000         65.500
   2.  STORAGE    128.00        150.00         240.00
   3.  DEMAND     6.4286        8.0000         UNBOUNDED

*7* VARIABLE TRANSITIONS RESULTING FROM RANGING OBJECTIVE CONTRIB
        VARIABLE         LOWER BOUND              UPPER BOUND
                       VAR. IN   VAR. OUT      VAR. IN    VAR. OUT
   1. SIX-OZ          CHAMP     SIX-OZ        +STORAGE   +DEMAND
   2. TEN-OZ          +STORAGE  +DEMAND       CHAMP      SIX-OZ
   3. CHAMP                                   CHAMP      SIX-OZ

*8* VARIABLE TRANSITIONS RESULTING FROM RANGING RHS RHS
        CONSTRNT         LOWER BOUND              UPPER BOUND
                       VAR. IN   VAR. OUT      VAR. IN    VAR. OUT
   1. PROD-HR          +STORAGE  SIX-OZ        CHAMP      +DEMAND
   2. STORAGE          CHAMP     +DEMAND       +STORAGE   SIX-OZ
   3. DEMAND           CHAMP     +DEMAND

OUTPUT OPTION? NO
```

Figure 3.1   Solution of the custom-molder example.

3.7   SIMULTANEOUS VARIATIONS WITHIN THE RANGES

Until now we have described the sensitivity of an optimal solution in the form of ranges on the objective-function coefficients and righthand-side values. These ranges were shown to be valid for changes in *one* objective-function coefficient or righthand-side value, while the remaining problem data are held fixed. It is then natural to ask what can be said about simultaneously changing *more* than one coefficient or value within the ranges. In the event that simultaneous changes are made *only* in objective-function coefficients of *nonbasic* variables and righthand-side values of *nonbinding* constraints within their appropriate ranges, the basis will remain unchanged. Unfortunately, it is not true, in general, that when the simultaneous variations within the ranges involve basic variables or binding constraints the basis will remain unchanged. However, for both the ranges on the objective-function coefficients when basic variables are involved, and the righthand-side values when binding constraints are involved, there is a conservative bound on these simultaneous changes that we refer to as the ''100 percent rule.''

Let us consider first, simultaneous changes in the righthand-side values involving binding constraints for which the basis, and therefore the shadow prices and reduced costs, remain unchanged. The righthand-side ranges, as discussed in Section 3.4, give the range over which *one* particular righthand-side value may be varied, with all other problem data being held fixed, such that the basis remains unchanged. As we have indicated, it is not true that simultaneous changes of more than one righthand-side value within these ranges will leave the optimal basis unchanged. However, it turns out that if these simultaneous changes are made in such a way that the sum of the fractions of allowable range utilized by these changes is less than or equal to one, the optimal basis will be unchanged.

Let us consider our custom-molder example, and make simultaneous changes in the binding production and storage capacity constraints. The righthand-side range on production capacity is 37.5 to 65.5 hundred hours, with the current value being 60. The righthand-side range on storage capacity is 128 to 240 hundred cubic feet, with the current value being 150. Although it is not true that the optimal basis remains unchanged for simultaneous changes in the current righthand-side values anywhere within these ranges, it is true that the optimal basis remains unchanged for any simultaneous change that is a weighted combination of values within these ranges. Figure 3.2 illustrates this situation. The horizontal and vertical lines in the figure are the ranges for production and storage capacity, respectively. The four-sided figure includes all weighted combinations of these ranges and is the space over which simultaneous variations in the values of production and storage capacity can be made while still ensuring that the basis remains unchanged. If we consider moving from the current righthand-side values of 60 and 150 to b_1^{new} and b_2^{new} respectively, where $b_1^{\text{new}} \leq 60$ and $b_2^{\text{new}} \geq 150$, we can ensure that the basis remains unchanged if

$$\frac{60 - b_1^{\text{new}}}{60 - 37.5} + \frac{b_2^{\text{new}} - 150}{240 - 150} \leq 1.$$

As long as the sum of the fractions formed by the ratio of the change to the maximum possible change in that direction is less than or equal to one, the basis remains unchanged. Hence, we have the 100 percent rule. Since the basis remains unchanged, the shadow prices and reduced costs also are unchanged.

A similar situation exists in the case of simultaneous variations in the objective-function coefficients when basic variables are involved. It is not true that the basis remains unchanged for simultaneous changes in these coefficients anywhere within their individual ranges. However, it is true that the optimal basis remains unchanged for any simultaneous change in the objective-function coefficients that is a weighted combination of the values within these ranges. If, for example, we were to increase all cost coefficients simultaneously, the new optimal basis would remain unchanged so long as the new values c_1^{new}, c_2^{new}, and c_3^{new} satisfy:

$$\frac{c_1^{\text{new}} - 5}{5.4 - 5} + \frac{c_2^{\text{new}} - 4.5}{6.5 - 4.5} + \frac{c_3^{\text{new}} - 6}{6.5714 - 6} \leq 1.$$

Again, the sum of the fractions formed by the ratio of the change in the coefficient to the maximum possible change in that direction must be less than or equal to one.
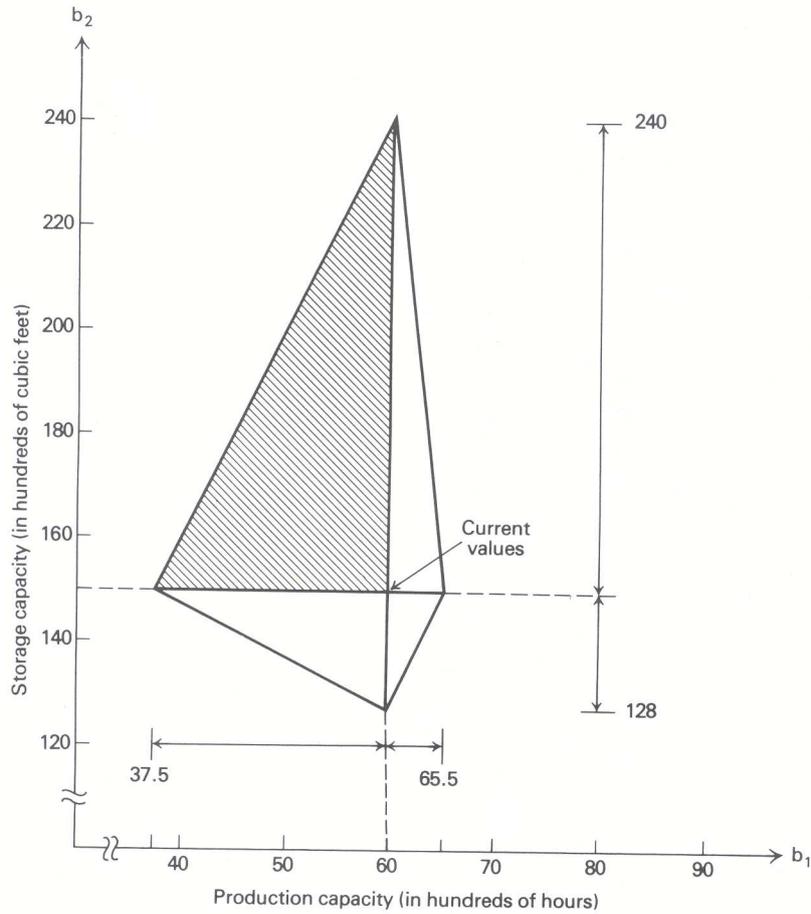
Figure 3.2   Simultaneous variation of righthand-side values.

General Discussion

Let us analyze the 100 percent rule more formally. If we first consider simultaneous variations in the righthand-side values, the 100 percent rule states that the basis remains unchanged so long as the sum of the fractions, corresponding to the percent of maximum change in each direction, is less than or equal to one. To see that this must indeed be true, we look at weighted combinations of the solution for the current values and the solutions corresponding to particular boundaries of the ranges. In Fig. 3.2, the shaded area contains all weighted combinations of the current values, the lower bound on b_1, and the upper bound on b_2.

Let x_j^0, $j = 1, 2, \ldots, n$, be an optimal solution to the given linear program, and let a_{ij} be the coefficients of the initial tableau corresponding to the basic variables. Then,

$$\sum_{j_B} a_{ij} x_j^0 = b_i \qquad (i = 1, 2, \ldots, m), \tag{19}$$

where j_B indicates that the sum is taken only over basic variables. Further, let x_j^k, $j = 1, 2, \ldots, n$, be the optimal solution at either the upper or lower limit of the range when changing b_k alone, depending on the direction of the variation being considered. Since the basis remains unchanged for these variations, these solutions satisfy:

$$\sum_{j_B} a_{ij} x_j^k = b_i' = \begin{cases} b_k + \Delta b_k^{\max} & \text{for } i = k, \\ b_i & \text{for } i \neq k. \end{cases} \tag{20}$$

It is now easy to show that any nonnegative, weighted combination of these solutions must be an optimal feasible solution to the problem with simultaneous variations. Let λ_0 be the weight associated with the

optimal solution corresponding to the current values, Eq. (19), and let λ_k be the weight associated with the optimal solution corresponding to the variation in b_k in Eq. (20). Consider all solutions that are nonnegative weighted combinations of these solutions, such that:

$$\sum_{k=0}^{m} \lambda_k = 1. \tag{21}$$

The corresponding weighted solution must be nonnegative; that is,

$$x_j = \sum_{k=0}^{m} \lambda_k x_j^k \geq 0 \qquad (j = 1, 2, \ldots, n), \tag{22}$$

since both λ_k and x_j^k are nonnegative. By multiplying the ith constraint of Eq. (19) by λ_0 and the ith constraint of Eq. (20) by λ_k and adding, we have

$$\sum_{k=0}^{m} \lambda_k \left(\sum_{j_B} a_{ij} x_j^k \right) = \sum_{k=0}^{m} \lambda_k b_i'.$$

Since the righthand-side reduces to:

$$\sum_{k=0}^{m} \lambda_k b_i' = \sum_{k \neq i} \lambda_k b_k + \lambda_i (b_i + \Delta b_i^{\max}) = b_i + \lambda_i \Delta b_i^{\max},$$

we can rewrite this expression by changing the order of summation as:

$$\sum_{j_B} a_{ij} \left(\sum_{k=0}^{m} \lambda_k x_j^k \right) = b_i + \lambda_i \Delta b_i^{\max} \qquad (i = 1, 2, \ldots, m). \tag{23}$$

Expressions (22) and (23) together show that the weighted solution x_j, $j = 1, 2, \ldots, n$, is a feasible solution to the righthand-side variations indicated in Eq. (20) and has the same basis as the optimal solution corresponding to the current values in Eq. (19). This solution must also be optimal, since the operations carried out do not change the objective-function coefficients.

Hence, the basis remains optimal so long as the sum of the weights is one, as in Eq. (21). However, this is equivalent to requiring that the weights λ_k, $k = 1, 2, \ldots, m$, corresponding to the solutions associated with the ranges, while *excluding* the solution associated with the current values, satisfy:

$$\sum_{k=1}^{m} \lambda_k \leq 1. \tag{24}$$

The weight λ_0 on the solution corresponding to the current values is then determined from Eq. (21). Expression (24) can be seen to be the 100 percent rule by defining:

$$\Delta b_k \equiv \lambda_k \Delta b_k^{\max}$$

or, equivalently,

$$\lambda_k = \frac{\Delta b_k}{\Delta b_k^{\max}}$$

which, when substituted in Eq. (24) yields:

$$\sum_{k=1}^{m} \frac{\Delta b_k}{\Delta b_k^{\max}} \leq 1. \tag{25}$$

Since it was required that $\lambda_k \geq 0$, Δb_k and Δb_k^{max} must have the same sign. Hence, the fractions in the 100 percent rule are the ratio of the actual change in a particular direction to the maximum possible change in that direction.

A similar argument can be made to establish the 100 percent rule for variations in the objective-function coefficients. The argument will not be given, but the form of the rule is:

$$\sum_{k=1}^{n} \frac{\Delta c_k}{\Delta c_k^{\text{max}}} \leq 1. \tag{26}$$

3.8 PARAMETRIC PROGRAMMING

In the sensitivity analysis discussed thus far, we have restricted our presentation to changes in the problem data that can be made without changing the optimal basis. Consequently, what we have been able to say is fairly conservative. We did go so far as to indicate the variable that would enter the basis and the variable that would leave the basis when a boundary of a range was encountered. Further, in the case of alternative optimal solutions and alternative optimal shadow prices, the indicated pivot was completed at least far enough to exhibit the particular alternative. One important point in these discussions was the ease with which we could determine the pivot to be performed at a boundary of a range. This seems to indicate that it is relatively easy to make systematic calculations beyond the indicated objective-function or righthand-side ranges. This, in fact, is the case; and the procedure by which these systematic calculations are made is called *parametric programming*.

Preview

Consider once again our custom-molder example, and suppose that we are about to negotiate a contract for storage space and are interested in knowing the optimal value of the objective function for all values of storage capacity. We know from Section 3.2 that the shadow price on storage capacity is $\frac{1}{35}$ hundred dollars per hundred cubic feet, and that this shadow price holds over the range of 128 to 240 hundred cubic feet. As long as we are negotiating within this range, we know the worth of an additional hundred cubic feet of storage capacity. We further know that if we go above a storage capacity of 240, x_5, the slack variable in the storage-capacity constraint, enters the basis and x_1, the production of six-ounce juice glasses, leaves the basis. Since slack in storage capacity exists beyond 240, the shadow price beyond 240 must be zero. If we go below 128, we know that x_3, production of champagne glasses, will enter the basis and the slack in six-ounce juice-glass demand leaves the basis. However, we do not know the shadow price on storage capacity below 128.

To determine the shadow price on storage capacity below 128, we need to perform the indicated pivot and exhibit the new canonical form. Once we have the new canonical form, we immediately know the new shadow prices and can easily compute the new righthand-side ranges such that these shadow prices remain unchanged. The new shadow price on storage capacity turns out to be $\frac{3}{55}$ and this holds over the range 95 to 128. We can continue in this manner until the optimal value of the objective function for all possible values of storage capacity is determined.

Since the shadow price on a particular constraint is the change in the optimal value of the objective function per unit increase in the righthand-side value of that constraint, the optimal value of the objective function within some range must be a linear function of the righthand-side value with a slope equal to the corresponding shadow price. In Fig. 3.3, the optimal value of the objective function is plotted versus the available storage capacity. Note that this curve consists of a number of straight-line segments. The slope of each straight-line segment is equal to the shadow price on storage capacity, and the corresponding interval is the righthand-side range for storage capacity. For example, the slope of this curve is $\frac{1}{35}$ over the interval 128 to 240 hundred cubic feet.

In order to determine the curve given in Fig. 3.3, we essentially need to compute the optimal solution for all possible values of storage capacity. It is intuitive that this may be done efficiently since the breakpoints
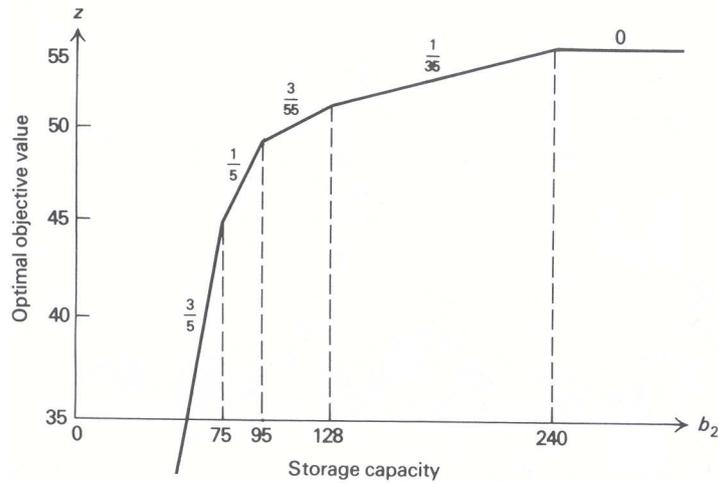
Figure 3.3   Righthand-side parametrics.
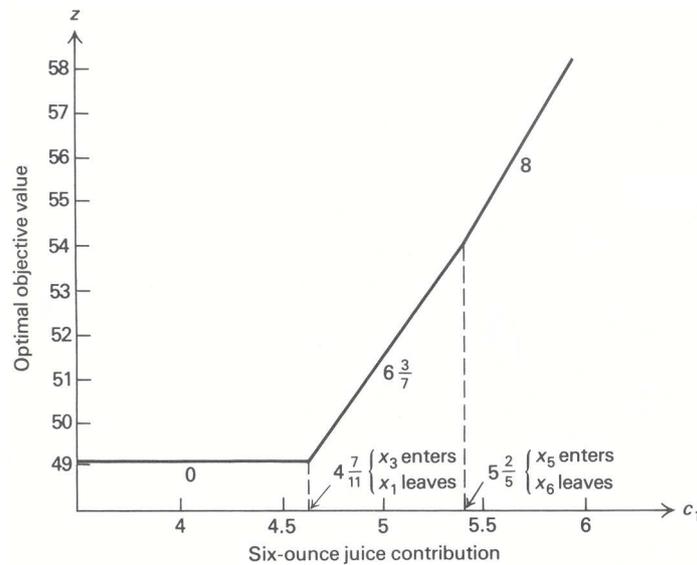
Figure 3.4   Objective parametrics.

in the curve correspond to changes in the basis, and the optimal solution need only be calculated at these breakpoints. From a given solution the next breakpoint may be determined, since this is the same as computing a righthand-side range.

Now let us fix the storage capacity again at 150 hundred cubic feet and consider solving our custom-molder example as a function of the contribution from six-ounce juice glasses. Since the basis will remain unchanged for variations within the objective-function ranges, we might expect results similar to those obtained by varying a righthand-side value. In Fig. 3.4, the optimal value of the objective function is plotted versus the contribution from six-ounce juice glasses. Note that this curve also consists of a number of straight-line segments. These segments correspond to ranges on the objective-function coefficient of six-ounce juice glasses such that the optimal basis does not change. Since the curve is a function of the contribution from production of six-ounce juice glasses, and the basis remains unchanged over each interval, the slope of the curve is given by the value of x_1 in the appropriate range.

In general then, it is straightforward to find the optimal solution to a linear program as a function of any *one* parameter: hence, the name *parametric programming*. In the above two examples, we first found

the optimal solution as a function of the righthand-side value b_2, and then found the optimal solution as a function of the objective-function coefficient c_1. The procedure used in these examples is easily generalized to include simultaneous variation of more than one coefficient, as long as the variation is made a function of *one* parameter.

Righthand-Side Parametrics

To illustrate the general procedure in detail, consider the trailer-production problem introduced in Chapter 2. That example included two constraints, a limit of 24 days/month on metalworking capacity and a limit of 60 days/month on woodworking capacity. Suppose that, by reallocating floor space and manpower in the workshop, we can exchange any number of days of woodworking capacity for the same number of days of metalworking capacity. After such an exchange, the capacities will become $(24 + \theta)$ days/month for metalworking capacity and $(60 - \theta)$ days/month for woodworking. The initial tableau for this problem is then given in Tableau 11. What is the optimal contribution to overhead that the firm can obtain for each value of θ? In particular, what value of θ provides the greatest contribution to overhead?

Tableau 11

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_4	$24 + \theta$	$\frac{1}{2}$	2	1	1	
x_5	$60 - \theta$	1	2	4		1
$(-z)$	0	+6	+14	+13		

We can answer these questions by performing parametric righthand-side analysis, starting with the final linear-programming tableau that was determined in Chapter 2 and is repeated here as Tableau 12.

Tableau 12

	Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
at $\theta = -7\frac{1}{5}$ ←	x_1	$36 + 5\theta$	1	6		4	$\widebar{1}$
at $\theta = 4$ ←	x_3	$6 - \frac{3}{2}\theta$		$\widebar{1}$	1	-1	$\frac{1}{2}$
	$(-z)$	$-294 - 10\frac{1}{2}\theta$		-9		-11	$-\frac{1}{2}$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{at } \theta = 4 & \text{at } \theta = -7\frac{1}{5} \end{array}$$

In Tableau 12 we have introduced the changes in the current values of the basic variables implied by the parametric variation. These are obtained in the usual way. Increasing metalworking capacity by θ units is equivalent to setting its slack variable $x_4 = -\theta$, and decreasing woodworking capacity by θ is equivalent to setting its slack variable $x_5 = +\theta$. Making these substitutions simultaneously in the tableau and moving the parameter θ to the righthand side gives the current values specified in Tableau 12.

Since the reduced costs are not affected by the value of θ, we see that the basis remains optimal in the final tableau so long as the basic variables x_1 and x_2 remain nonnegative. Hence,

$$x_1 = 36 + 5\theta \geq 0 \qquad \left(\text{that is, } \theta \geq -7\tfrac{1}{5}\right),$$
$$x_2 = 6 - \tfrac{3}{2}\theta \geq 0 \qquad (\text{that is, } \theta \leq 4),$$

which implies that the optimal contribution is given by

$$z = 294 + 10\tfrac{1}{2}\theta \qquad \text{for} -7\tfrac{1}{5} \leq \theta \leq 4.$$

At $\theta = 4$, variable x_3 becomes zero, while at $\theta = -7\frac{1}{5}$, variable x_1 becomes zero, and in each case a pivot operation is indicated. As we saw in Section 5 on variable transitions associated with righthand-side ranges,

Tableau 13

	Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
	x_5	$-36 - 5\theta$	-1	-6		-4	1
at $\theta = -24$ ←	x_3	$24 + \theta$	$\frac{1}{2}$	2	1	1	
	$(-z)$	$-312 - 13\theta$	$-\frac{1}{2}$	-12		-13	

the variable to be introduced at these values of θ is determined by pivoting on the negative coefficient in the appropriate pivot row r that gives the minimum ratio as follows:

$$\frac{\bar{c}_s}{\bar{a}_{rs}} = \underset{j}{\text{Min}} \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} \,\middle|\, \bar{a}_{rj} < 0 \right\}.$$

At $\theta = -7\frac{1}{5}$, only variable x_5 has a negative coefficient in the pivot row, so we pivot to introduce x_5 into the basis in place of x_1. The result is given in Tableau 13.

Tableau 13 is optimal for:

$$x_5 = -36 - 5\theta \geq 0 \qquad \text{(that is, } \theta \leq -7\tfrac{1}{5}\text{)},$$

and

$$x_3 = \quad 24 - \quad \theta \geq 0 \qquad \text{(that is, } \theta \geq -24\text{)},$$

and the optimal contribution is then:

$$z = 312 + 13\theta \qquad \text{for} -24 \leq \theta \leq -7\tfrac{1}{5}.$$

At $\theta = -24$, x_3 is zero and becomes a candidate to drop from the basis. We would like to replace x_3 in the basis with another decision variable for $\theta < -24$. We cannot perform any such pivot, however, because there is no negative constraint coefficient in the pivot row. The row reads:

$$\tfrac{1}{2}x_1 + 2x_2 + x_3 + x_4 = 24 + \theta.$$

For $\theta < -24$, the righthand side is negative and the constraint cannot be satisfied by the nonnegative variables x_1, x_2, x_3, and x_4. Consequently, the problem is infeasible for $\theta < -24$. This observation reflects the obvious fact that the capacity $(24 + \theta)$ in the first constraint of the original problem formulation becomes negative for $\theta < -24$ and so the constraint is infeasible.

Having now investigated the problem behavior for $\theta \leq 4$, let us return to Tableau 12 with $\theta = 4$. At this value, x_2 replaces x_3 in the basis. Performing the pivot gives Tableau 14.

Tableau 14

	Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
at $\theta = 18$ ←	x_1	$72 - 4\theta$	1		6	-2	2
	x_2	$-6 + \frac{3}{2}\theta$		1	-1	1	$-\frac{1}{2}$
	$(-z)$	$-348 + 3\theta$			-9	-2	-5

The basic variables are nonegative in Tableau 14 for $4 \leq \theta \leq 18$ with optimal objective value $z = 348 - 3\theta$. At $\theta = 18$, we must perform another pivot to introduce x_4 into the basis in place of x_1, giving Tableau 15.

The basic variables are nonnegative in Tableau 15 for $18 \leq \theta \leq 60$ with optimal objective value $z = 420 - 7\theta$. At $\theta = 60$, no variable can be introduced into the basis for the basic variable x_2, which reaches

Tableau 15

	Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
	x_4	$2\theta - 36$	$-\frac{1}{2}$		-3	1	-1
at $\theta = 60 \leftarrow$	x_2	$30 - \frac{1}{2}\theta$	$\frac{1}{2}$	1	2		$\frac{1}{2}$
	$(-z)$	$-420 + 7\theta$	-1		-15		-7
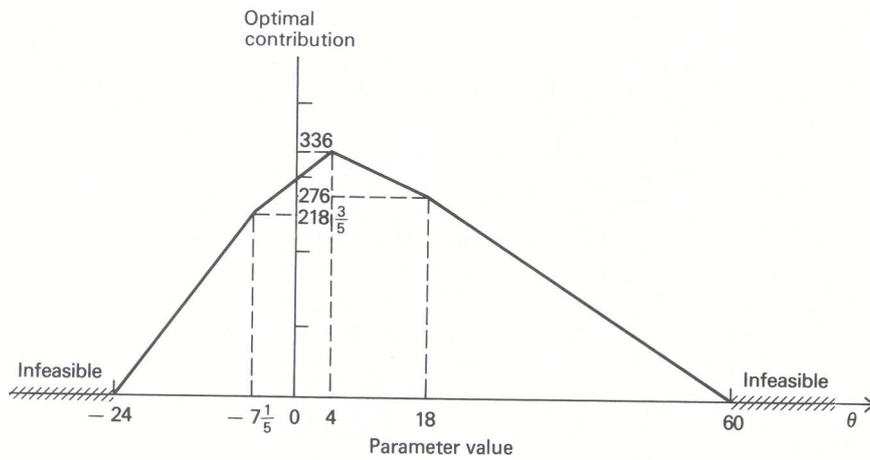
Figure 3.5 Parametric righthand-side analysis.

zero. This is caused by an infeasibility in the original problem formulation, since the woodworking capacity $60 - \theta$ becomes negative for any value of θ in excess of 60.

By collecting the various pieces of the optimal contribution as a function of the parameter θ, we determine the graph given in Fig. 3.5. The highest contribution that can be obtained is \$33,600 per month, which occurs at $\theta = 4$, with

$$24 + \theta = 28 \text{ days/month of metalworking capacity,}$$

and

$$60 - \theta = 56 \text{ days/month of woodworking capacity.}$$

By exchanging 4 days of capacity, we can increase contribution by more than 15 percent from its original value of \$29,400 permonth. This increase must be weighed against the costs incurred in the changeover.

Objective-Function Parametrics

To illustrate parametric programming of the objective-function coefficients, we will consider the trailer-production example once again. However, this time we will keep the capacities unchanged and vary the contribution coefficients in the objective function. Suppose that our trailer manufacturer is entering a period of extreme price competition and is considering changing his prices in such a way that his contribution would become:

$$c_1 = 6 + \theta,$$
$$c_2 = 14 + 2\theta,$$
$$c_3 = 13 + 2\theta.$$

How does the optimal production strategy change when the contribution is reduced (i.e., as θ becomes more and more negative)?

Tableau 16

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_4	24	$\frac{1}{2}$	2	1	1	
x_5	60	1	2	4		1
$(-z)$	0	$6+\theta$	$14+2\theta$	$13+2\theta$		

The initial tableau for this example is then Tableau 16, and the final tableau, assuming $\theta = 0$, is Tableau 17.

Tableau 17

	Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
←	x_1	36	1	⑥		4	-1
	x_3	6		-1	1	-1	$\frac{1}{2}$
	$(-z)$	$-294-48\theta$		$(-9-2\theta)$		$(-11-2\theta)$	$-\frac{1}{2}$

$$\uparrow$$
$$\text{at } \theta = -\tfrac{9}{2}$$

As θ is allowed to decrease, the current solution remains optimal so long as $\theta \geq -\frac{9}{2}$. At this value of θ, the reduced cost of x_2 becomes zero, indicating that there may be alternative optimal solutions. In fact, since a pivot is possible in the first row, we can determine an alternative optimal solution where x_2 enters the basis and x_1 leaves. The result is given in Tableau 18.

Tableau 18

	Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
	x_2	6	$\frac{1}{6}$	1		$\frac{2}{3}$	$-\frac{1}{6}$
←	x_3	12	$\frac{1}{6}$		1	$-\frac{1}{3}$	⑤$\frac{1}{3}$
	$(-z)$	$-240-36\theta$	$(\frac{3}{2}+\frac{1}{3}\theta)$			$(-5-\frac{2}{3}\theta)$	$(-2-\frac{1}{3}\theta)$

$$\uparrow$$
$$\text{at } \theta = -6$$

The new basis, consisting of variables x_2 and x_3, remains optimal so long as $-6 \leq \theta \leq -4\frac{1}{2}$. Once $\theta = -6$, the reduced cost of x_5 is zero, so that x_5 becomes a candidate to enter the basis. A pivot can be performed so that x_5 replaces x_3 in the basis. The result is given in Tableau 19.

Tableau 19

	Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
←	x_2	12	$\frac{1}{4}$	1	$\frac{1}{2}$	⑤$\frac{1}{2}$	
	x_5	36	$\frac{1}{2}$		3	-1	1
	$(-z)$	$-168-24\theta$	$(\frac{5}{2}+\frac{1}{2}\theta)$		$(6+\theta)$	$(-7-\theta)$	

$$\uparrow$$
$$\text{at } \theta = -7$$

The new basis, consisting of the variables x_2, and x_5, remains optimal so long as $-7 \leq \theta \leq -6$. At

$\theta = -7$, x_4 enters the basis, replacing x_2, and the resulting tableau is identical to Tableau 16 that has the slack basis x_4 and x_5 and so will not be repeated here.
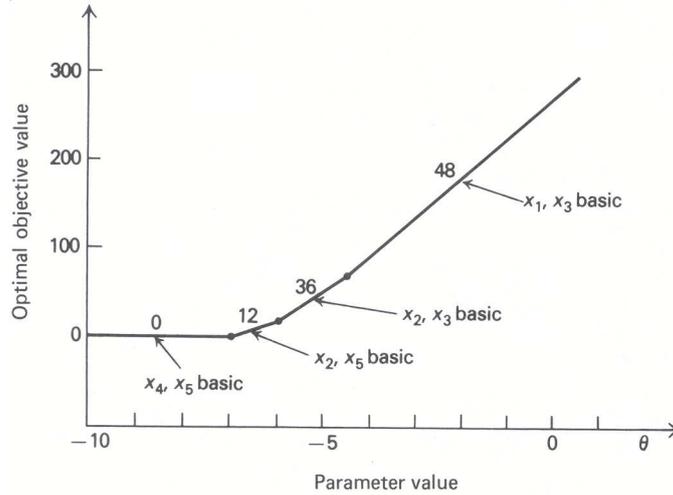
Figure 3.6   Parametric objective-function analysis.

In Fig. 3.6, we plot the optimal value of the objective function as a function of θ. As θ is decreased to $-4\frac{1}{2}$, the contributions of the three products become $1\frac{1}{2}$, 5, and 4, respectively, and x_1 is replaced in the basis by x_2. As we continue to decrease θ to -6, the contributions become 0, 2, and 1, and x_2 replaces x_3 in the basis. It is interesting to note that x_2 is not in the optimal basis with $\theta = 0$, but eventually is the only product produced. At $\theta = 0$, even though x_2 has the highest contribution, it is not produced because a combination of x_1 and x_3 is a better match with the productive resources available. As the relative contributions are reduced, x_2 eventually becomes more attractive than x_1 and finally is the only product manufactured.

The algorithm for objective-function parametrics is clearly straightforward to apply and consists of computing a sequence of optimal bases by the primal simplex method. At each stage, a range is computed on the parameter θ such that the current basis remains optimal. The variable transition at the boundary of this range is easily determined, and one pivot operation determines a new basis that is optimal for some adjacent range.

EXERCISES

1. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.1.xls`] Outdoors, Inc. has, as one of its product lines, lawn furniture. They currently have three items in that line: a lawn chair, a standard bench, and a table. These products are produced in a two-step manufacturing process involving the tube-bending department and the welding department. The time required by each item in each department is as follows:

	Product			Present
	Lawn chair	Bench	Table	capacity
Tube bending	1.2	1.7	1.2	1000
Welding	0.8	0	2.3	1200

The contribution that Outdoors, Inc. receives from the manufacture and sale of one unit of each product is $3 for a chair, $3 for a bench, and $5 for a table.

The company is trying to plan its production mix for the current selling season. It feels that it can sell any number it produces, but unfortunately production is further limited by available material, because of a prolonged

strike. The company currently has on hand 2000 lbs. of tubing. The three products require the following amounts of this tubing: 2 lbs. per chair, 3 lbs. per bench, and 4.5 lbs. per table.

In order to determine the optimal product mix, the production manager has formulated the linear program shown in Fig. E3.1 and obtained the results shown in Fig. E3.2.

	Chair	Bench	Table	Relation	Limit
D-Tube	1.2	1.7	1.2	≤	1000
D-Weld	0.8	0	2.3	≤	1200
TSuply	2.0	3.0	4.5	≤	2000
Contrib.	3.00	3.00	5.00	=	z (max)

Figure E3.1   Formulation of Outdoors, Inc.

```
TITLE: OUTDOORS
PROCEED, DISPLAY, OR REJECT? PROCEED

MAXIMIZE OR MINIMIZE? MAX

OPTIMAL SOLUTION FOUND.
     CONTRIB.      2766.67

OUTPUT OPTION? EXTEND

ALL ITEMS NOT LISTED IN SECTIONS 1 - 4 HAVE THE VALUE ZERO.

*1* DECISION VARIABLES
   1. CHAIR      700.000
   3. TABLE      133.333

*2* SLACK(+) AND SURPLUS(-) IN CONSTRAINTS
   2. +D-WELD    333.333

*3* SHADOW PRICES FOR CONSTRAINTS
   1. D-TUBE     1.16667
   3. TSUPLY     .800000

*4* REDUCED COSTS FOR DECISION VARIABLES
   2. BENCH      -1.38333

*5* RANGES ON COEFFICIENTS OF OBJECTIVE CONTRIB.
      VARIABLE   LOWER BOUND   CURRENT VALUE   UPPER BOUND
   1. CHAIR        2.2222         3.0000         5.0000
   2. BENCH       UNBOUNDED       3.0000         4.3833
   3. TABLE        3.0000         5.0000         6.7500

*6* RANGES ON VALUES OF RIGHT-HAND-SIDE LIMIT
      CONSTRNT   LOWER BOUND   CURRENT VALUE   UPPER BOUND
   1.  D-TUBE      533.33        1000.0         1200.0
   2.  D-WELD      866.67        1200.0        UNBOUNDED
   3.  TSUPLY      1666.7        2000.0         2555.6

OUTPUT OPTION? NO
```

Figure E3.2   Solution of Outdoors, Inc.

a) What is the optimal production mix? What contribution can the firm anticipate by producing this mix?
b) What is the value of one unit more of tube-bending time? of welding time? of metal tubing?
c) A local distributor has offered to sell Outdoors, Inc. some additional metal tubing for $0.60/lb. Should Outdoors buy it? If yes, how much would the firm's contribution increase if they bought 500 lbs. and used it in an optimal fashion?
d) If Outdoors, Inc. feels that it must produce at least 100 benches to round out its product line, what effect will that have on its contribution?

e) The R&D department has been redesigning the bench to make it more profitable. The new design will require 1.1 hours of tube-bending time, 2.0 hours of welding time, and 2.0 lbs. of metal tubing. If it can sell one unit of this bench with a unit contribution of $3, what effect will it have on overall contribution?

f) Marketing has suggested a new patio awning that would require 1.8 hours of tube-bending time, 0.5 hours of welding time, and 1.3 lbs. of metal tubing. What contribution must this new product have to make it attractive to produce this season?

g) Outdoors, Inc. has a chance to sell some of its capacity in tube bending at cost + $1.50/hour. If it sells 200 hours at that price, how will this affect contribution?

h) If the contribution on chairs were to decrease to $2.50, what would be the optimal production mix and what contribution would this production plan give?

2. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.2.xls`] A commercial printing firm is trying to determine the best mix of printing jobs it should seek, given its current capacity constraints in its four capital-intensive departments: typesetting, camera, pressroom, and bindery. It has classified its commercial work into three classes: A, B, and C, each requiring different amounts of time in the four major departments.

The production requirements in hours per unit of product are as follows:

| | Class of work | | |
Department	A	B	C
Typesetting	0	2	3
Camera	3	1	3
Pressroom	3	6	2
Bindery	5	4	0

Assuming these units of work are produced using regular time, the contribution to overhead and profit is $200 for each unit of Class A work, $300 for each unit of Class B work, and $100 for each unit of Class C work.

The firm currently has the following regular-time capacity available in each department for the next time period: typesetting, 40 hours; camera, 60 hours; pressroom, 200 hours; bindery, 160 hours. In addition to this regular time, the firm could utilize an overtime shift in typesetting, which would make available an additional 35 hours in that department. The premium for this overtime (i.e., incremental costs in addition to regular time) would be $4/hour.

Since the firm wants to find the optimal job mix for its equipment, management assumes it can sell all it produces. However, to satisfy long-established customers, management decides to produce at least 10 units of each class of work in each time period.

Assuming that the firm wants to maximize its contribution to profit and overhead, we can formulate the above situation as a linear program, as follows:

Decision variables:

X_{AR} = Number of units of Class A work produced on regular time;

X_{BR} = Number of units of Class B work produced on regular time;

X_{CR} = Number of units of Class C work produced on regular time;

X_{BO} = Number of units of Class B work produced on overtime typesetting;

X_{CO} = Number of units of Class C work produced on overtime typesetting.

Objective function:

Maximize $z = 200X_{AR} + 300X_{BR} + 100X_{CR} + 292X_{BO} + 88X_{CO}$,

Constraints:

Regular Typesetting		$2X_{BR}+$	$3X_{CR}$		$\leq$ 40,
Overtime Typesetting				$2X_{BO} + 3X_{CO}$	$\leq$ 35,
Camera	$3X_{AR} +$	$X_{BR} +$	$3X_{CR} +$	$X_{BO} + 3X_{CO}$	$\leq$ 60,
Pressroom	$3X_{AR} +$	$6X_{BR} +$	$2X_{CR} +$	$6X_{BO} + 2X_{CO}$	$\leq$ 200,
Bindery	$5X_{AR} +$	$4X_{BR} +$		$4X_{BO}$	$\leq$ 160,
Class A—minimum	X_{AR}				$\geq$ 10,
Class B—minimum		X_{BR}	$+$	X_{BO}	$\geq$ 10,
Class C—minimum			X_{CR}	$+$ X_{CO}	$\geq$ 10,

$$X_{AR} \geq 0, \quad X_{BR} \geq 0, \quad X_{CR} \geq 0, \quad X_{BO} \geq 0, \quad X_{CO} \geq 0.$$

The solution of the firm's linear programming model is given in Fig. E3.3.

a) What is the optimal production mix?
b) Is there any unused production capacity?
c) Is this a unique optimum? Why?
d) Why is the shadow price of regular typesetting different from the shadow price of overtime typesetting?
e) If the printing firm has a chance to sell a new type of work that requires 0 hours of typesetting, 2 hours of camera, 2 hours of pressroom, and 1 hour of bindery, what contribution is required to make it attractive?
f) Suppose that both the regular and overtime typesetting capacity are reduced by 4 hours. How does the solution change? (*Hint*: Does the basis change in this situation?)

3. Jean-Pierre Leveque has recently been named the Minister of International Trade for the new nation of New France. In connection with this position, he has decided that the welfare of the country (and his performance) could best be served by maximizing the net dollar value of the country's exports for the coming year. (The net dollar value of exports is defined as exports *less* the cost of all materials imported by the country.)

   The area that now constitutes New France has traditionally made three products for export: steel, heavy machinery, and trucks. For the coming year, Jean-Pierre feels that they could sell all that they could produce of these three items at existing world market prices of $900/unit for steel, $2500/unit for machinery, and $3000/unit for trucks.

   In order to produce one unit of steel with the country's existing technology, it takes 0.05 units of machinery, 0.08 units of trucks, two units of ore purchased on the world market for $100/unit, and other imported materials costing $100. In addition, it takes .5 man-years of labor to produce each unit of steel. The steel mills of New France have a maximum usable capacity of 300,000 units/year.

   To produce one unit of machinery requires .75 units of steel, 0.12 units of trucks, and 5 man-years of labor. In addition, $150 of materials must be imported for each unit of machinery produced. The practical capacity of the country's machinery plants is 50,000 units/year.

   In order to produce one unit of trucks, it takes one unit of steel, 0.10 units of machinery, three man-years of labor, and $500 worth of imported materials. Existing truck capacity is 550,000 units/year.

   The total manpower available for production of steel, machinery, and trucks is 1,200,000 men/year.

   To help Jean-Pierre in his planning, he had one of his staff formulate the model shown in Fig. E3.4 and solved in Fig. E3.5. Referring to these two exhibits, he has asked you to help him with the following questions:

a) What is the optimal production and export mix for New France, based on Fig.E3.5? What would be the net dollar value of exports under this plan?
b) What do the first three constraint equations (STEEL, MACHIN, and TRUCK) represent? Why are they equality constraints?
c) The optimal solution suggests that New France produce 50,000 units of machinery. How are those units to be utilized during the year?
d) What would happen to the value of net exports if the world market price of steel increased to $1225/unit and the country chose to export one unit of steel?

```
TITLE: COMMERCIAL PRINTING
PROCEED, DISPLAY, OR REJECT? PROCEED

MAXIMIZE OR MINIMIZE? MAX

OPTIMAL SOLUTION FOUND.
     CONTRIB      10110.0

OUTPUT OPTION? EXTENDED

ALL ITEMS NOT LISTED IN SECTIONS 1 - 4 HAVE THE VALUE ZERO.

*1* DECISION VARIABLES
  1. CLASS-AR  12.5000
  2. CLASS-BR  20.0000
  4. CLASS-BO  2.50000
  5. CLASS-CO  10.0000

*2* SLACK(+) AND SURPLUS(-) IN CONSTRAINTS
  4. +PRESS    7.50000
  5. +BINDERY  7.50000
  6. -CLA-MIN  2.50000
  7. -CLB-MIN  12.5000

*3* SHADOW PRICES FOR CONSTRAINTS
  1. TYPE-REG  116.667
  2. TYPE-OVT  112.667
  3. CAMERA     66.6667
  8. CLC-MIN  -450.000

*4* REDUCED COSTS FOR DECISION VARIABLES
  3. CLASS-CR  .000000

*5* RANGES ON COEFFICIENTS OF OBJECTIVE CONTRIB
     VARIABLE  LOWER BOUND  CURRENT VALUE  UPPER BOUND
  1. CLASS-AR   .19073E-05     200.00        876.00
  2. CLASS-BR   300.00         300.00       UNBOUNDED
  3. CLASS-CR  UNBOUNDED       100.00        100.00
  4. CLASS-BO   66.667         292.00        292.00
  5. CLASS-CO   88.000         88.000        538.00

*6* RANGES ON VALUES OF RIGHT-HAND-SIDE CAPACITY
     CONSTRNT  LOWER BOUND  CURRENT VALUE  UPPER BOUND
  1.  TYPE-REG   15.000        40.000        43.000
  2.  TYPE-OVT   30.000        35.000        38.000
  3.  CAMERA     82.500        90.000        94.500
  4.  PRESS      192.50        200.00       UNBOUNDED
  5.  BINDERY    152.50        160.00       UNBOUNDED
  6.  CLA-MIN   UNBOUNDED      10.000        12.500
  7.  CLB-MIN   UNBOUNDED      10.000        22.500
  8.  CLC-MIN    9.1176        10.000        11.667

OUTPUT OPTION? NO
```

Figure E3.3   Solution for commericial printing firm.

e) If New France wants to identify other products it can profitably produce and export, what characteristics should those products have?

f) There is a chance that Jean-Pierre may have $500,000 to spend on expanding capacity. If this investment will buy 500 units of truck capacity, 1000 units of machine capacity, or 300 units of steel capacity, what would be the best investment?

g) If the world market price of the imported materials needed to produce one unit of trucks were to increase by $400, what would be the optimal export mix for New France, and what would be the dollar value of their net exports?

Variables:

X_1 = Steel production for export (EXSTEE),

X_2 = Machinery production for export (EXMACH),

X_3 = Truck production for export (EXTRUC),

X_4 = Total steel production (TOSTEE),

X_5 = Total machinery production (TOMACH),

X_6 = Total truck production (TOTRUC).

Constraints:

Steel output	(STEEL)
Machinery ouput	(MACHIN)
Truck output	(TRUCK)
Steel capacity	(CAPSTE)
Machinery capacity	(CAPMAC)
Truck capacity	(CAPTRU)
Manpower available	(AVAMAN)

```
TITLE: NEW FRANCE INTERNATIONAL TRADE
PROCEED, DISPLAY, OR REJECT? DISPLAY


OBJECTIVES:

               EXSTEE     EXMACH     EXTRUC     TOSTEE     TOMACH     TOTRUC

EXPORTS        900.0      2500.      3000.      -300.0     -150.0     -500.0


CONSTRAINTS:

               EXSTEE     EXMACH     EXTRUC     TOSTEE     TOMACH     TOTRUC
               RELATION   CAPACITY

STEEL          -1.000     .0000      .0000      1.000      -.7500     -1.000
               EQ         .0000
MACHIN         .0000      -1.000     .0000      -.5000E-01 1.000      -.1000
               EQ         .0000
TRUCK          .0000      .0000      -1.000     -.8000E-01 -.1200     1.000
               EQ         .0000
CAPSTE         .0000      .0000      .0000      1.000      .0000      .0000
               LE         .3000E+06
CAPMAC         .0000      .0000      .0000      .0000      1.000      .0000
               LE         .5000E+05
CAPTRU         .0000      .0000      .0000      .0000      .0000      1.000
               LE         .5500E+06
AVAMAN         .0000      .0000      .0000      .5000      5.000      3.000
               LE         .1200E+07
```

Figure E3.4   Formulation of optimal production and export for New France.

h) The Minister of Defense has recently come to Jean-Pierre and said that he would like to stockpile (inventory) an additional 10,000 units of steel during the coming year. How will this change the constraint equation STEEL, and what impact will it have on net dollar exports?

i) A government R&D group has recently come to Jean-Pierre with a new product, Product X, that can be produced for export with 1.5 man-years of labor and 0.3 units of machinery for each unit produced. What must Product X sell for on the world market to make it attractive for production?

j) How does this particular formulation deal with existing inventories at the start of the year and any desired inventories at the end of the year?

4. Another member of Jean-Pierre's staff has presented an alternative formulation of New France's planning problem

```
TITLE: NEW FRANCE INTERNATIONAL TRADE
PROCEED, DISPLAY, OR REJECT? PROCEED

MAXIMIZE OR MINIMIZE? MAX

OPTIMAL SOLUTION FOUND.
     EXPORTS      0.490625E+09

OUTPUT OPTION? EXTENDED

ALL ITEMS NOT LISTED IN SECTIONS 1 - 4 HAVE THE VALUE ZERO.

*1* DECISION VARIABLES
  2. EXMACH     8750.00
  3. EXTRUC     232500.
  4. TOSTEE     300000.
  5. TOMACH     50000.0
  6. TOTRUC     262500.

*2* SLACK(+) AND SURPLUS(-) IN CONSTRAINTS
  6. +CAPTRU    287500.
  7. +AVAMAN    12500.0

*3* SHADOW PRICES FOR CONSTRAINTS
  1. STEEL      -2250.00
  2. MACHIN     -2500.00
  3. TRUCK      -3000.00
  4. CAPSTE      1585.00
  5. CAPMAC      302.500

*4* REDUCED COSTS FOR DECISION VARIABLES
  1. EXSTEE     -1350.00

*5* RANGES ON COEFFICIENTS OF OBJECTIVE EXPORTS
       VARIABLE   LOWER BOUND   CURRENT VALUE   UPPER BOUND
  1. EXSTEE       UNBOUNDED        900.00         2250.0
  2. EXMACH        2218.6          2500.0         13067.
  3. EXTRUC        1650.0          3000.0         3347.7
  4. TOSTEE       -1885.0          -300.00        UNBOUNDED
  5. TOMACH       -452.50          -150.00        UNBOUNDED
  6. TOTRUC       -1850.0          -500.00        -96.667

*6* RANGES ON VALUES OF RIGHT-HAND-SIDE CAPACITY
       CONSTRNT   LOWER BOUND   CURRENT VALUE   UPPER BOUND
  1.  STEEL       -4166.7        .00000         .23250E+06
  2.  MACHIN      UNBOUNDED      .00000         8750.0
  3.  TRUCK       UNBOUNDED      .00000         .23250E+06
  4.  CAPSTE      47283.         .30000E+06     .30357E+06
  5.  CAPMAC      41860.         50000.         54545.
  6.  CAPTRU      .26250E+06     .55000E+06     UNBOUNDED
  7.  AVAMAN      .11875E+07     .12000E+07     UNBOUNDED

OUTPUT OPTION? NO
```

Figure E3.5   Solution of New France model.

as described in Exercise 3, which involves only three variables. This formulation is as follows:

$$Y_1 = \text{Total steel production},$$
$$Y_2 = \text{Total machinery production},$$
$$Y_3 = \text{Total truck production}.$$

$$\text{Maximize } z = 900(Y_1 - 0.75Y_2 - Y_3) - 300Y_1 + 2500(Y_2 - 0.05Y_1 - 0.10Y_3)$$
$$-150Y_2 + 3000(Y_3 - 0.80Y_1 - 0.12Y_2) - 500Y_3,$$

subject to:

$$Y_1 \leq 300{,}000,$$
$$Y_2 \leq 50{,}000,$$
$$Y_3 \leq 550{,}000,$$

$$0.5Y_1 + 5Y_2 + 3Y_3 \leq 1{,}200{,}000, \qquad Y_1, Y_2, Y_3 \geq 0.$$

a) Is this formulation equivalent to the one presented in Fig. E3.4 of Exercise 3? How would the optimal solution here compare with that found in Fig. E3.5?

b) If we had the optimal solution to this formulation in terms of total production, how would we find the optimal exports of each product?

c) What assumption does this formulation make about the quantities and prices of products that can be exported and imported?

d) New France is considering the production of automobiles. It will take 0.5 units of steel, 0.05 units of machinery, and 2.0 man-years to produce one unit of automobiles. Imported materials for this unit will cost $250 and the finished product will sell on world markets at a price of $2000. Each automobile produced will use up .75 units of the country's limited truck capacity. How would you alter this formulation to take the production of automobiles into account?

5. Returning to the original problem formulation shown in Fig. E3.4 of Exercise 3, Jean-Pierre feels that, with existing uncertainties in international currencies, there is some chance that the New France dollar will be revalued upward relative to world markets. If this happened, the cost of imported materials would go down by the same percent as the devaluation, and the market price of the country's exports would go up by the same percent as the revaluation. Assuming that the country can always sell all it wishes to export how much of a revaluation could occur before the optimal solution in Fig. E3.5 would change?

6. After paying its monthly bills, a family has $100 left over to spend on leisure. The family enjoys two types of activities: (i) eating out; and (ii) other entertainment, such as seeing movies and sporting events. The family has determined the relative value (utility) of these two activities—each dollar spend on eating out provides 1.2 units of value for each dollar spent on other entertainment. Suppose that the family wishes to determine how to spend this money to maximize the value of its expenditures, but that no more than $70 can be spent on eating out and no more than $50 on other entertainment (if desired, part of the money can be saved). The family also would like to know:

a) How the total value of its expenditures for leisure would change if there were only $99 to spend.

b) How the total value would change if $75 could be spend on eating out.

c) Whether it would *save* any money if each dollar of savings would provide 1.1 units of value for each dollar spent on other entertainment.

Formulate the family's spending decision as a linear program. Sketch the feasible region, and answer each of the above questions graphically. Then solve the linear program by the simplex method, identify shadow prices on the constraints, and answer each of these questions, using the shadow prices.

7. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.7.xls`] Consider the linear program:

Maximize $z = x_1 + 3x_2$,

subject to:

$$
\begin{array}{rcll}
x_1 + & x_2 \leq 8 & \text{(resource 1)}, \\
-x_1 + & x_2 \leq 4 & \text{(resource 2)}, \\
x_1 & \leq 6 & \text{(resource 3)}, \\
x_1 \geq 0, & x_2 \geq 0.
\end{array}
$$

a) Determine graphically:

   i) the optimal solution;
   ii) the shadow prices on the three constraints;
   iii) the range on the objective coefficient of each variable, holding the coefficient of the other variable at its current value, for which the solution to part (i) remains optimal;
   iv) the range on the availability of each resource, holding the availability of the other resources at their current values, for which the shadow prices in part (ii) remain optimal.

  b) Answer the same question, using the following optimal tableau determined by the simplex method:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_1	2	1		$\frac{1}{2}$	$-\frac{1}{2}$	
x_2	6		1	$\frac{1}{2}$	$\frac{1}{2}$	
x_5	4			$-\frac{1}{2}$	$\frac{1}{2}$	1
$(-z)$	-20			-2	-1	

8. Determine the variable transitions for the previous problem, when each objective coefficient or righthand-side value is varied by itself to the point where the optimal basis no longer remains optimal. Carry out this analysis using the optimal tableau, and interpret the transitions graphically on a sketch of the feasible region.

9. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.9.xls`] A wood-products company produces four types of household furniture. Three basic operations are involved: cutting, sanding, and finishing. The plant's capacity is such that there is a limit of 900 machine hours for cutting, 800 hours for sanding, and 480 hours for finishing. The firm's objective is to maximize profits. The initial tableau is as follows:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_5	480	2	8	4	2	1		
x_6	800	5	4	8	5		1	
x_7	900	7	8	3	5			1
$(-z)$	0	$+90$	$+160$	$+40$	$+100$			

Using the simplex algorithm, the final tableau is found to be:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_2	25	0	1	$\frac{1}{8}$		$\frac{5}{32}$	$-\frac{1}{16}$	
x_4	140	1		$\frac{3}{2}$	1	$-\frac{1}{8}$	$\frac{1}{4}$	
x_7	0	2		$-\frac{11}{2}$		$-\frac{5}{8}$	$-\frac{3}{4}$	1
$(-z)$	$-18,000$	-10		-130		$-12\frac{1}{2}$	-15	

  a) What are the shadow prices on each of the constraints?
  b) What profit for x_3 would justify its production?
  c) What are the limits on sanding capacity that will allow the present basic variables to stay in the optimal solution?
  d) Suppose management had to decide whether or not to introduce a new product requiring 20 hours of cutting, 3 hours of sanding, and 2 hours of finishing, with an expected profit of $120. Should the product be produced?
  e) If another saw to perform cutting can be rented for $10/hour, should it be procured? What about a finisher at the same price? If either is rented, what will be the gain from the first hour's use?

10. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.10.xls`] The Reclamation Machining Company makes nuts and bolts from scrap material supplied from the waste products of three steel-using

firms. For each 100 pounds of scrap material provided by firm A, 10 cases of nuts and 4 cases of bolts can be made, with a contribution of $46. 100 pounds from firm B results in 6 cases of nuts, 10 cases of bolts, and a contribution of $57. Use of 100 pounds of firm C's material will produce 12 cases of nuts, 8 of bolts, and a contribution of $60. Assuming Reclamation can sell only a maximum of 62 cases of nuts and 60 of bolts, the final tableau for a linear-programming solution of this problem is as follows:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_1	3.421	1		0.947	0.132	−0.079
x_2	4.632		1	0.421	−0.052	0.132
$(-z)$	−421.368			−7.759	−3.053	−3.868

a) What is the optimal solution?
b) Is the solution unique? Why?
c) For each of the three sources, determine the interval of contribution for which the solution in part (a) remains optimal.
d) What are the shadow prices associated with the optimal solution in (a), and what do they mean?
e) Give an interval for each sales limitation such that the shadow prices in (d) remain unchanged.

11. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.11.xls`] Consider the linear program:

$$\text{Maximize } z = 2x_1 + x_2 + 10x_3,$$

subject to:

$$x_1 - x_2 + 3x_3 \quad\quad = 10,$$
$$x_2 + x_3 + x_4 = 6,$$
$$x_j \geq 0 \quad (j = 1, 2, 3, 4).$$

The optimal tableau is:

Basic variables	Current values	x_1	x_2	x_3	x_4
x_3	4	$\frac{1}{4}$		1	$\frac{1}{4}$
x_2	2	$-\frac{1}{4}$	1		$\frac{3}{4}$
$(-z)$	−24	$-\frac{1}{4}$			$-3\frac{1}{4}$

a) What are the optimal shadow prices for the two constraints? Can the optimal shadow price for the first constraint be determined easily from the final objective coefficient for x_1? (*Hint*: The initial problem formulation is in canonical form if the objective coefficient of x_1 is changed to $0x_1$.)
b) Suppose that the initial righthand-side coefficient of the first constraint is changed to $10 + \delta$. For what values of δ do x_2 and x_3 form an optimal basis for the problem?

12. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.12.xls`] Consider the following linear program:

$$\text{Minimize } z = 2x_i + x_2 + 2x_3 - 3x_4,$$

subject to:

$$8x_1 - 4x_2 - x_3 + 3x_4 \leq 10,$$
$$2x_1 + 3x_2 + x_3 - x_4 \leq 7,$$
$$- 2x_2 - x_3 + 4x_4 \leq 12,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.$$

After several iterations of the simplex algorithm, the following tableau has been determined:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_5	11	10		$\frac{1}{2}$		1	1	$-\frac{1}{2}$
x_2	4	$\frac{4}{5}$	1	$\frac{3}{10}$			$\frac{2}{5}$	$\frac{1}{10}$
x_4	5	$\frac{2}{5}$		$-\frac{1}{10}$	1		$\frac{1}{5}$	$\frac{3}{10}$
$(-z)$	-11	$\frac{12}{5}$		$\frac{7}{5}$			$\frac{1}{5}$	$\frac{4}{5}$

a) What are the values of the decision variables? How do you know this is the optimal solution?
b) For each nonbasic variable of the solution, find the interval for its objective-function coefficient such that the solution remains optimal.
c) Do the same for each basis variable.
d) Determine the interval for each righthand-side value such that the optimal basis determined remains unchanged. What does this imply about the shadow prices? reduced costs?

13. A cast-iron foundry is required to produce 1000 lbs. of castings containing at least 0.35 percent manganese and not more than 3.2 percent silicon. Three types of pig iron are available in unlimited amounts, with the following properties:

	Type of pig iron		
	A	B	C
Silicon	4%	1%	5%
Manganese	0.35%	0.4%	0.3%
Cost/1000 lbs.	$28	$30	$20

Assuming that pig iron is melted with other materials to produce cast iron, a linear-programming formulation that minimizes cost is as follows:

$$\text{Minimize } z = 28x_1 + 30x_2 + 20x_3,$$

subject to:

$$
\begin{aligned}
4x_1 + x_2 + 5x_3 &\le 3.2 && (\text{lb. Si} \times 10), \\
3.5x_1 + 4x_2 + 3x_3 &\ge 3.5 && (\text{lb. Mn}), \\
x_1 + x_2 + x_3 &= 1 && (\text{lb.} \times 10^3),
\end{aligned}
$$

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Initial tableau

					Surplus	Slack	Artificial	
Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	v_1	v_2
x_5	3.2	4	1	5	0	1		
v_1	3.5	3.5	4	3	-1		1	
v_2	1	1	1	1	0			1
$(-z)$	0	28	30	20	0			

Final tableau

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	v_1	v_2
x_5	0.2	1			-4	1	4	-17
x_3	0.5	0.5	1		-1		1	-3
x_2	0.5	0.5		1	1		-1	4
$(-z)$	-25	3			10		-10	10

a) At what cost does pig type A become a candidate for entry into the optimal basis? What activity would it replace?

b) How much can we afford to pay for pure manganese to add to the melt?

c) How much can the manganese requirement be reduced without changing the basis? What are the values of the other basic variables when this happens?

d) How much can the cost of pig type B change without changing the optimal basis? What are the new basic variables when such a change occurs?

e) How can the final tableau be optimal if the reduced cost of v_1 is -10?

14. [Excel spreadsheet available at http://web.mit.edu/15.053/www/Exer3.14.xls] The Classic Stone Cutter Company produces four types of stone sculptures: figures, figurines, free forms, and statues. Each product requires the following hours of work for cutting and chiseling stone and polishing the final product:

	Type of product			
Operation	Figures	Figurines	Free Forms	Statues
Cutting	30	5	45	60
Chiseling	20	8	60	30
Polishing	0	20	0	120
Contribution/unit	$280	$40	$500	$510

The last row in the above table specifies the contribution to overhead for each product.

   Classic's current work force has production capacity to allocate 300 hours to cutting, 180 hours to chiseling, and 300 hours to polishing in any week. Based upon these limitations, it finds its weekly production schedule from the linear-programming solution given by the following tableau.

Basic variables	Current values	Figures x_1	Figurines x_2	Forms x_3	Statues x_4	Cutting slack, hours x_5	Chiseling slack, hours x_6	Polishing slack, hours x_7
Statues	2		$-\frac{7}{15}$	-6	1	$\frac{1}{15}$	$-\frac{1}{10}$	
Figures	6	1	$\frac{11}{10}$	$\frac{15}{2}$		$-\frac{1}{10}$	$\frac{1}{5}$	
Slack	60		76	360		-8	12	1
$-$Contrib.	-2700		-30	-70		-6	-5	

a) Determine a range on the cutting capacity such that the current solution remains optimal.

b) Busts have the following characteristics:

$$\begin{array}{ll} \text{Cutting} & \text{15 hrs.} \\ \text{Chiseling} & \text{10 hrs.} \\ \text{Polishing} & \text{20 hrs.} \\ \text{Contribution/unit} & \$240 \end{array}$$

Should Classic maintain its present line or expand into busts?

c) Classic can buy 5 hours of cutting capacity and 5 hours of chiseling capacity from an outside contractor at a total cost of $75. Should Classic make the purchase or not?

d) By how much does the contribution from free forms have to be increased to make free forms profitable to produce?

e) Give a range on the contribution from figures such that the current solution remains optimal. What activities enter the basis at the bounds of this range?

15. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.15.xls`] The Concrete Products Corporation has the capability of producing four types of concrete blocks. Each block must be subjected to four processes: batch mixing, mold vibrating, inspection, and yard drying. The plant manager desires to maximize profits during the next month. During the upcoming thirty days, he has 800 machine hours available on the batch mixer, 1000 hours on the mold vibrator, and 340 man-hours of inspection time. Yard-drying time is unconstrained. The production director has formulated his problem as a linear program with the following initial tableau:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_5	800	1	2	10	16	1		
x_6	1000	1.5	2	4	5		1	
x_7	340	0.5	0.6	1	2			1
$(-z)$	0	8	14	30	50			

where x_1, x_2, x_3, x_4 represent the number of pallets of the four types of blocks. After solving by the simplex method, the final tableau is:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_2	200		1	+11	19	15	−1	
x_1	400	1		−12	−22	−2	2	
x_7	20			0.4	1.6	0.1	−0.4	1
$(-z)$	6000			−28	−40	−5	−2	

a) By how much must the profit on a pallet of number 3 blocks be increased before it would be profitable to manufacture them?

b) What minimum profit on x_2 must be realized so that it remains in the production schedule?

c) If the 800 machine-hours capacity on the batch mixer is uncertain, for what range of machine hours will it remain feasible to produce blocks 1 and 2?

d) A competitor located next door has offered the manager additional batch-mixing time at a rate of $4.00 per hour. Should he accept this offer?

e) The owner has approached the manager with a thought about producing a new type of concrete block that would require 4 hours of batch mixing, 4 hours of molding, and 1 hour of inspection per pallet. What should be the profit per pallet if block number 5 is to be included in the optimal schedule?

16. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.16.xls`] The linear-programming program:

$$\text{Maximize } z = \qquad 4x_4 + 2x_5 - 3x_6,$$

subject to:

$$
\begin{aligned}
x_1 \phantom{{}+{}} &+ x_4 - x_5 + 4x_6 = 2, \\
x_2 &+ x_4 + x_5 - 2x_6 = 6, \\
x_3 &- 2x_4 + x_5 - 3x_6 = 6, \\
x_j &\geq 0 \qquad (j = 1, 2, \ldots, 6)
\end{aligned}
$$

has an optimal canonical form given by:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_4	4	$\frac{1}{2}$	$\frac{1}{2}$		1		1
x_5	2	$-\frac{1}{2}$	$\frac{1}{2}$			1	-3
x_3	12	$\frac{3}{2}$	$\frac{1}{2}$	1			2
$(-z)$	-20	-1	-3				-1

Answer the following questions. All parts are independent and refer to changes in the *original* problem formulation as stated above. That is, do not consider parts (a), (b), and (c) together, but refer to the original problem formulation in each part.

a) Suppose that the objective function changes to

$$z = 4x_4 + 2x_5 + 0x_6.$$

Find the new optimal solution. Is the solution to the new problem unique? Explain your answer.

b) Consider the parametric-programming problem with the objective function:

$$\text{Maximize } z = (4 + 2\theta)x_4 + (2 - 4\theta)x_5 + (-3 + \theta)x_6.$$

For what values $\underline{\theta} \leq \theta \leq \overline{\theta}$ of θ do the variables x_4, x_5, and x_3 remain in the optimal basis (e.g., if $\underline{\theta} = -1$ and $\overline{\theta} = 2$, then the interval is $-1 \leq \theta \leq 2$). What are the variable transitions at $\theta = \underline{\theta}$ and $\theta = \overline{\theta}$?

c) Consider the parametric programming problem with the righthand sides

$$2 + 4\theta,$$
$$6 - 2\theta,$$
$$6 - \theta.$$

For what values $\underline{\theta} \leq \theta \leq \overline{\theta}$ of θ do the variables x_4, x_5, and x_3 remain in the optimal basis? What are the variable transitions at $\theta = \underline{\theta}$ and $\theta = \overline{\theta}$?

17. The Massachusetts Electric Company is planning additions to its mix of nuclear and conventional electric power plants to cover a ten-year horizon. It measures the output of plants in terms of equivalent conventional plants. Production of electricity must satisfy the state's additional requirements over the planning horizon for guaranteed power (kW), peak power (kW), and annual energy (MWh). The state's additional requirements, and the contribution of each type of plant to these requirements, are as follows:

	Requirements		
	Guaranteed power (kW)	Peak power (kW)	Annual energy (MWh)
Conventional	1	3	1
Nuclear	1	1	4
Additional requirements	20	30	40

The costs for each type of plant include fixed investment costs and variable operating costs per year. The conventional plants have very high operating costs, since they burn fossil fuel, while the nuclear plants have very high investment costs. These costs are as follows:

Type of plant	Investment costs (millions)	Annual operating costs (millions)
Conventional	$ 30	$20
Nuclear	100	5

For simplicity, we will assume that the annual operating costs are an infinite stream discounted at a rate of $(1/(1+r))$ per year. The present value of this infinite stream of operating costs is:

$$PV = \sum_{n=1}^{\infty} \left(\frac{1}{1+r}\right)^n c = \left[\frac{1}{1 - [1/(1+r)]} - 1\right] c = \frac{1}{r} c$$

and the term $1/r$ is sometimes referred to as the *coefficient of capitalization.*

a) Formulate a linear program to decide what mix of conventional and nuclear power plants to build, assuming that you must meet or exceed the state's three requirements and you want to minimize the *total* investment plus discounted operating costs.

b) There has been a debate at Massachusetts Electric as to what discount rate r to use in making this calculation. Some have argued for a low "social" rate of about 2 or 3 percent, while others have argued for the private "cost of capital" rate of from 10 to 15 percent. Graphically determine the optimal solution for all values of the discount rate r, to aid in settling this question. Comment on the implications of the choice of discount rate.

c) Another difficulty is that there may not be sufficient investment funds to make the decisions independently of an investment limitation. Assuming the "social" discount rate of 2 percent, find the optimal solution for all conceivable budget levels for total investment. What is the impact of limiting the budget for investment?

18. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.18.xls`] Consider the parametric-programming problem:

$$\text{Maximize } z = (-3 + 3\theta)x_1 + (1 - 2\theta)x_2,$$

subject to:

$$\begin{aligned}
-2x_1 + x_2 &\leq 2, \\
x_1 - 2x_2 &\leq 2, \\
x_1 - x_2 &\leq 4, \\
x_1 \geq 0, \quad x_2 &\geq 0.
\end{aligned}$$

Letting x_3, x_4, and x_5 be slack variables for the constraints, we write the optimal canonical form at $\theta = 0$ as:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_2	2	−2	1	1		
x_4	6	−3		2	1	
x_5	6	−1		1		1
$(-z)$	−2	−1		−1		

a) Place the objective function in this tableau in canonical form for values of θ other than 0. For what values of θ is this canonical form optimal?

b) What are the variable transitions when this canonical form is no longer optimal? Does the problem become unbounded at either of the transition points?

c) Use the parametric programming algorithm to find the optimal solution for all values of θ. Plot the optimal objective value as a function of θ.

d) Graph the feasible region and interpret the parametric algorithm on the graph.

19. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.19.xls`] Consider the following parametric linear program:

$$z^*(\theta) = \text{Max } x_1 + 2x_2,$$

subject to:

$$\begin{aligned}
2x_1 + x_2 &\leq 14, \\
x_1 + x_2 &\leq 10, \\
x_1 &\leq 1 + 2\theta, \\
x_2 &\leq 8 - \theta, \\
x_1 \geq 0, \quad x_2 &\geq 0.
\end{aligned}$$

a) Graph the feasible region and indicate the optimal solution for $\theta = 0$, $\theta = 1$, and $\theta = 2$.

b) For what values of θ is this problem feasible?

c) From the graphs in part (a), determine the optimal solution for all values of θ. Graph the optimal objective value $z^*(\theta)$.

d) Starting with the optimal canonical form given below for $\theta = 0$, use the parametric simplex algorithm to solve for all values of θ.

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_3	$4 - 3\theta$			1		-2	-1
x_4	$1 - \theta$				1	-1	-1
x_1	$1 + 2\theta$	1				1	
x_2	$8 - \theta$		1				1
$(-z)$	-17					-1	-2

20. Consider the linear program:

$$\text{Minimize } z = -10x_1 + 16x_2 - x,$$

subject to:

$$x_1 - 2x_2 + x_3 \le 2 + 2\theta,$$
$$x_1 - x_2 \le 4 + \theta,$$
$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0,$$

where θ is a parameter.

a) For $\theta = 0$

   i) Solve the linear program.

   ii) What are the optimal shadow prices?

   iii) Suppose that the constant on the righthand side of the second inequality is changed from 4 to 6. What is the new optimal value of z?

b) For what values of θ does the optimal basis to part a(i) remain optimal?

c) Solve the linear program for all values of θ.

21. When discussing parametric analysis in Section 3.8, we considered reallocating floor space for a trailer-production problem to trade off woodworking capacity for metalworking capacity. The trade-off was governed by the parameter θ in the following tableau:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	
x_4	$24 + \theta$	$\frac{1}{2}$	2	1	1		Woodworking capacity
x_5	$60 - \theta$	1	2	4		1	Metalworking capacity
$(-z)$	0	6	14	13			

We found that the following tableau was optimal for values of θ in the range $0 \le \theta \le 4$:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_1	$35 + 5\theta$	1	6		4	-1
x_3	$6 - \frac{3}{2}\theta$		-1	1	-1	$\frac{1}{2}$
$(-z)$	$-294 - 10\frac{1}{2}\theta$		-9		-11	$-\frac{1}{2}$
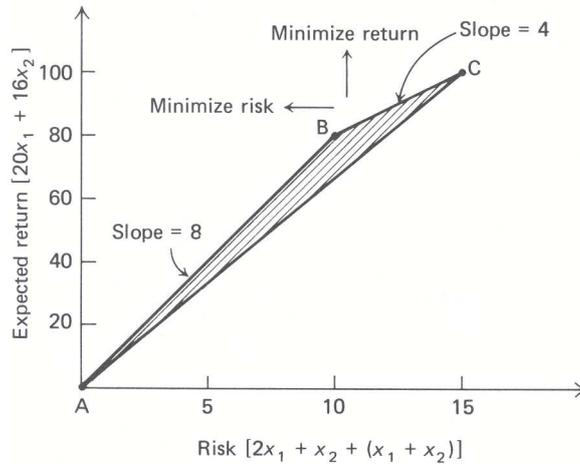
Figure E3.6 Risk-return trade-off.

The optimal solution at $\theta = 4$ is $x_1 = 56$, $x_2 = x_3 = x_4 = x_5 = 0$, and $z = 336$. At $\theta = 5$, we found that the optimal basic variables are $x_1 = 52$, $x_2 = 1.5$, and that the optimal objective value is $z = 348 - 3(5) = 333$. Therefore as θ increases from 4 to 5,

$$\Delta z = 333 - 336 = -3.$$

The previous optimal tableau tells us that for $0 \le \theta \le 4$, woodworking capacity is worth \$11/day and metalworking capacity is worth \$0.50/day. Increasing θ from 4 to 5 increases woodworking capacity by one day. Using the prices \$11/day and \$0.50/day, the change in the optimal objective value should be:

$$\Delta z = 11(1) + 0.50(-1) = 10.50.$$

We have obtained two different values for the change in the optimal objective value, $\Delta z = -3$ and $\Delta z = 10.50$. Reconcile the difference between these values.

22. An investor has \$5000 and two potential investments. Let x_j for $j = 1$ and $j = 2$ denote her allocation to investment j in thousands of dollars. From historical data, investments 1 and 2 are known to have an expected annual return of 20 and 16 percent, respectively. Also, the total risk involved with investments 1 and 2, as measured by the variance of total return, is known to be given by $2x_1 + x_2 + (x_1 + x_2)$, so that risk increases with total investment $(x_1 + x_2)$ and with the amount of each individual investment. The investor would like to maximize her return and at the same time minimize her risk. Figure E3.6 illustrates the conflict between these two objectives. Point A in this figure corresponds to investing nothing, point B to investing all \$5000 in the second alternative, and point C to investing completely in the first alternative. Every other point in the shaded region of the figure corresponds to some other investment strategy; that is, to a feasible solution to the constraints:

$$\begin{aligned}
x_1 + x_2 &\le 5, \\
x_1 \ge 0, \quad x_2 &\ge 0.
\end{aligned} \tag{27}$$

To deal with the conflicting objectives, the investor decides to combine return and risk into a single objective function.

$$\begin{aligned}
\text{Objective} &= \text{Return} - \theta \, (\text{Risk}) \\
&= 20x_1 + 16x_2 - \theta[2x_1 + x_2 + (x_1 + x_2)].
\end{aligned} \tag{28}$$

She uses the parameter θ in this objective function to weigh the trade-off between the two objectives.

Because the ''most appropriate'' trade-off parameter is difficult to assess, the investor would like to maximize the objective function (28) subject to the constraints (27), for all values of θ, and then use secondary considerations, not captured in this simple model, to choose from the alternative optimal solutions.

a) Use parametric linear programming to solve the problem for all nonnegative values of θ.

b) Plot the optimal objective value, the expected return, and the risk as a function of θ.

c) Interpret the solutions on Fig. E3.6.

d) Suppose that the investor can save at 6% with no risk. For what values of θ will she save any money using the objective function (28)?

23. An alternative model for the investment problem discussed in the previous exercise is:

$$\text{Maximize } z = \quad 20x_1 + 16x_2,$$

subject to:

$$x_1 + \quad x_2 \quad\quad\quad\quad \leq 5,$$
$$2x_1 + \quad x_2 + (x_1 + x_2) \leq \gamma,$$

$$x_1 \geq 0, \quad\quad x_2 \geq 0.$$

In this model, the investor maximizes her expected return, constraining the risk that she is willing to incur by the parameter γ.

a) Use parametric linear programming to solve the problem for all nonnegative values of γ.

b) Plot the expected return and risk as a function of γ.

c) Interpret the solutions on Fig. E3.6 of Exercise 22.

24. The shadow-price concept has been central to our development of sensitivity analysis in this chapter. In this exercise, we consider how changing the initial problem formulation alters the development. Suppose that the initial formulation of the model is given by:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad\quad (i = 1, 2, \ldots, m),$$
$$x_j \geq 0 \quad\quad (j = 1, 2, \ldots, n).$$

Solving this problem by the simplex method determines the shadow prices for the constraints as $y_1, y_2, \ldots, y_m$.

a) Suppose that the first constraint were multiplied by 2 and stated instead as:

$$\sum_{j=1}^{n} (2a_{1j}) x_j \leq (2b_1).$$

Let $\hat{y}_1$ denote the value of the shadow price for this constraint. How is $\hat{y}_1$ related to y_1?

b) What happens to the value of the shadow prices if every coefficient c_j is multiplied by 2 in the original problem formulation?

c) Suppose that the first variable x_1 in the model is rescaled by a factor of 3. That is, let $x_1' = 3x_1$ and replace x_1 everywhere in the model by $(x_1'/3)$. Do the shadow prices change? Would it be possible for x_1' to appear in an optimal basis, if x_1 could not appear in an optimal basis of the original formulation for the problem?

d) Do the answers to parts (a), (b), and (c) change if the original problem is stated with all equality constraints:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad\quad (i = 1, 2, \ldots, m)?$$

25. [Excel spreadsheet available at `http://web.mit.edu/15.053/www/Exer3.25.xls`] Shortly after the beginning of the oil embargo, the Eastern District Director of the Government Office of Fuel Allocation was concerned that he would soon have to start issuing monthly allocations specifying the amounts of heating oil that each refinery in the district would send to each city to ensure that every city would receive its quota.

Since different refineries used a variety of alternative sources of crude oil, both foreign and domestic, the cost of products at each refinery varied considerably. Consequently, under current government regulations, the prices that could be charged for heating oil would vary from one refinery to another. To avoid political criticism, it was felt that, in making allocations of supplies from the refineries to the cities, it was essential to maintain a reasonably uniform average refinery price for each city. In fact, the Director felt that the average refinery price paid by any city should not be more than 3% above the overall average.

Another complication had arisen in recent months, since some of the emergency allocations of supplies to certain cities could not be delivered due to a shortage in tanker capacity. If deliveries of the allocated supplies were to be feasible, the limited transportation facilities on some of the shipping routes would have to be recognized. Finally, it would be most desirable to maintain distribution costs as low as possible under any allocation plan, if charges of government inefficiency were to be avoided.

Data for a simplified version of the problem, with only three refineries and four cities, are given in Fig. E3.7. The allocation problem can be formulated as the linear program shown in Fig. E3.8. The decision variables are defined as follows:

$A - n$ = Barrels of heating oil (in thousands) to be supplied from Refinery A to City n where $n = 1, 2, 3,$ or 4.

$B - n$ = Barrels of heating oil (in thousands) to be supplied from Refinery B to City n where $n = 1, 2, 3,$ or 4.

$C - n$ = Barrels of heating oil (in thousands) to be supplied by Refinery C to City n where $n = 1, 2, 3,$ or 4.

PMAX = A value higher than the average refinery price paid by any of the four cities.

There are four types of constraints that must be satisfied:

1. Supply constraints at each refinery.
2. Quota restrictions for each city.
3. Average price restrictions which impose an upper limit to the average refinery price which will be paid by any city.
4. Shipping capacity limits on some of the routes.

All the constraints are straightforward except possibly the average price reductions. To restrict the average refinery price paid by City 1 to be less than the variable PMAX, the following expression is used:

$$\frac{(10.53)(A - 1) + (9.39)(B - 1) + (12.43)(C - 1)}{55} \leq \text{PMAX}$$

or

$$0.1915(A - 1) + 0.1701(B - 1) + 0.2260(C - 1) - \text{PMAX} \leq 0.$$

Such a restriction is included for each city, to provide a uniform upper bound on average prices. The value of PMAX is limited by the prescribed maximum of $11.40 = 11.07 \times 1.03$ by the constraint

$$\text{PMAX} \leq 11.40.$$

The computer analysis of the linear programming model is given in Fig. E3.9

a) Determine the detailed allocations that might be used for December.
b) Evaluate the average refinery prices paid by each city under these allocations.
c) Determine the best way to utilize an additional tanker-truck capacity of 10,000 barrels per month.
d) Discuss the inefficiency in distribution costs resulting from the average refinery price restrictions.
e) Evaluate the applicability of the model in the context of a full allocation system.

Refinery	December availability (barrels)	Price at refinery ($ per barrel)
A. Norton	95,000	10.53
B. Chatham	63,000	9.39
C. Eastport	116,000	12.43
Average Refinery Price $11.07		

City	December quotas (barrels)
1. Westville	55,000
2. Bridgeton	73,000
3. Brookfield	105,000
4. Dorval	38,000

	Shipping costs ($ per barrel) City			
Refinery	1	2	3	4
A	0.10	0.16	0.32	0.28
B	0.20	0.34	0.30	0.12
C	0.34	0.38	0.22	0.18

	capacities (thousands of barrels per month) City			
Refinery	1	2	3	4
A	*	25	*	*
B	*	20	20	*
C	25	*	*	*

* No effective limit.

Figure E3.7   Data for heating oil allocation problem.

f) Discuss any general insights into the fuel allocation problem provided by this model.

26. The Krebs Wire Company is an intermediate processor that purchases uncoated wire in standard gauges and then applies various coatings according to customer specification. Krebs Wire has essentially two basic products—standard inexpensive plastic and the higher quality Teflon. The two coatings come in a variety of colors but these are changed easily by introducing different dyes into the basic coating liquid.

The production facilities at Krebs Wire consist of two independent wire trains, referred to as the Kolbert and Loomis trains. Both the standard plastic-coated and the quality Teflon-coated wire can be produced on either process train; however, production of Teflon-coated wire is a slower process due to drying requirements. The different production rates in tons per day are given below:

Process train	Plastic	Teflon
Kolbert	40 tons/day	35 tons/day
Loomis	50 tons/day	42 tons/day

It has been traditional at Krebs Wire to view production rates in terms of daily tonnage, as opposed to reels per day or other production measures. The respective contributions in dollars per day are:

Process train	Plastic	Teflon
Kolbert	525 $/day	546 $/day
Loomis	580 $/day	590 $/day

Planning at Krebs Wire is usually done on a monthly basis. However, since most employee vacations are scheduled over the two summer months, management feels that production planning for the two summer months should be combined to facilitate vacation scheduling. Each month the process trains must be shut down for scheduled maintenance, so that the total days available for production per month are as follows:

	$A-1$	$A-2$	$A-3$	$A-4$	$B-1$	$B-2$	$B-3$	$B-4$	$C-1$	$C-2$	$C-3$	$C-4$	PMAX	Relation	Righthand-side	
REF-A	1.0000	1.0000	1.0000	1.0000										$\leq$	95.0000	Supply
REF-B					1.0000	1.0000	1.0000	1.0000						$\leq$	63.0000	Constraints
REF-C									1.0000	1.0000	1.0000	1.0000		$\leq$	116.0000	
CIT-1	1.0000				1.0000				1.0000					=	55.0000	
CIT-2		1.0000				1.0000				1.0000				=	73.0000	Quota
CIT-3			1.0000				1.0000				1.0000			=	105.0000	Restrictions
CIT-4				1.0000				1.0000				1.0000		=	38.0000	
AP-CIT-1	0.1915				1.701				0.2260				−1.0000	$\leq$	0.0000	
AP-CIT-2		0.1442				0.1286				0.1703			−1.0000	$\leq$	0.0000	Average
AP-CIT-3			0.1003				0.0894				0.1184		−1.0000	$\leq$	0.0000	Price
AP-CIT-4				0.2771				0.2471				0.3271	−1.0000	$\leq$	0.0000	Restrictions
PRICELIM													1.0000	$\leq$	11.4000	
ROUTE-A2		1.0000												$\leq$	25.0000	Shipping
ROUTE-B2						1.0000								$\leq$	20.0000	Capacity
ROUTE-B3							1.0000							$\leq$	20.0000	Limits
ROUTE-C1									1.0000					$\leq$	25.0000	
Shipcost	0.1000	0.1600	0.3200	0.2800	0.2000	0.3400	0.3000	0.1200	0.3400	0.3800	0.2200	0.1800	0.0000	=	$z(min)$ *Objective*	

Figure E3.8   Formulation of the heating-oil allocation model.

Process train	July	August
Kolbert	26 days	26 days
Loomis	28 days	27 days

The scheduling process is further complicated by the fact that, over the two summer months, the total amount of time available for production is limited to 102 machine days due to vacation schedules.

The amounts of wire that the management feels it can sell in the coming two months are:

Product	July	August
Plastic	1200 tons	1400 tons
Teflon	800 tons	900 tons

Both types of wire may be stored for future delivery. Space is available in Krebs' own warehouse, which has a capacity of 20 tons. The inventory and carrying costs in dollars per ton for wire produced in July and delivered in August are:

Product	Inventory and carrying costs
Plastic	1.00 $/ton
Teflon	1.20 $/ton

Due to a commitment of warehouse capacity to other products in September, it is not possible to stock any wire in inventory at the end of August.

To help in planning production for the two summer months, Krebs Wire management has formulated and solved a linear program (see Figs. E3.10 and E3.11) in order to determine the production schedule that will maximize contribution from these products.

a) What does the output in Fig. E3.12 tell the production manager about the details of scheduling his machines?

b) There is the possibility that some employees might be persuaded to take their vacations in June or September. Should this be encouraged?

c) The solution in Fig. E3.12 suggests that Teflon should be made only on the Loomis train in July and only on the Kolbert train in August. Why?

d) Should Krebs Wire lease additional warehouse capacity at a cost of $2.00 above the inventory and carrying costs? If so, how would the optimal solution change?

e) The sales manager feels that future sales might be affected if the firm could not meet demand for plastic-coated wire in August. What, if anything, should be done?

f) One of Krebs' customers has requested a special run of twenty tons of corrosion-resistant wire to be delivered at the end of August. Krebs has made this product before and found that it can be produced only on the Loomis machine, due to technical restrictions. The Loomis machine can produce this special wire at a rate of 40 tons per day, and the customer will pay $12 per ton. Krebs cannot start production before the 1st of August due to a shortage of raw materials. Should the firm accept the order?

27. Consider the computer output for the Krebs Wire case in Exercise 26. What does the ''100 percent rule'' tell us in each of the following situations?

```
TITLE: FUEL ALLOCATION
PROCEED, DISPLAY, OR REJECT? PROCEED

MAXIMIZE OR MINIMIZE? MIN

OPTIMAL SOLUTION FOUND.
     SHIPCOST     61.6563

OUTPUT OPTION? USUAL

ALL ITEMS NOT LISTED IN SECTIONS 1 - 4 HAVE THE VALUE ZERO.

*1* DECISION VARIABLES
   1. A-1      54.4253
   2. A-2      15.6023
   3. A-3      24.9724
   5. B-1       .574694
   6. B-2      14.9803
   7. B-3      20.0000
   8. B-4      27.4450
  10. C-2      42.4174
  11. C-3      60.0276
  12. C-4      10.5550
  13. PMAX     11.4000

*2* SLACK(+) AND SURPLUS(-) IN CONSTRAINTS
   3. +REF-C     3.00000
  11. +AP-CIT-4 1.16580
  13. +ROUTE-A2 9.39768
  14. +ROUTE-B2 5.01967
  16. +ROUTE-C1 25.0000

*3* SHADOW PRICES FOR CONSTRAINTS
   1. REF-A     -.232518
   2. REF-B     -.600000E-01
   4. CIT-1      .341718
   5. CIT-2      .461679
   6. CIT-3     2.39515
   7. CIT-4      .180000
   8. AP-CIT-1 -.480411E-01
   9. AP-CIT-2 -.479616
  10. AP-CIT-3 -18.3712
  12. PRICELIM -18.8988
  15. ROUTE-B3 -.392764

*4* REDUCED COSTS FOR DECISION VARIABLES
   4. A-4        .332518
   9. C-1        .913941E-02

OUTPUT OPTION?
```

Figure E3.9   Solution of the allocation model. (*Continued on next page.*)

```
*5* RANGES ON COEFFICIENTS OF OBJECTIVE SHIPCOST
      VARIABLE  LOWER BOUND  CURRENT VALUE  UPPER BOUND
   1. A-1        .27482E-01      .10000        .10935
   2. A-2        .15065          .16000        .23252
   3. A-3        .74861E-01      .32000        .13630E+06
   4. A-4       -.52518E-01      .28000       UNBOUNDED
   5. B-1       UNBOUNDED        .20000        .27252
   6. B-2        .32000          .34000        .35494
   7. B-3       UNBOUNDED        .30000        .69276
   8. B-4        .10559          .12000        .14000
   9. C-1        .33086          .34000       UNBOUNDED
  10. C-2        .18615          .38000        .40000
  11. C-3       -.13630E+06      .22000        .55252
  12. C-4        .16000          .18000        .19441
  13. PMAX      UNBOUNDED        .00000       18.899


*6* RANGES ON VALUES OF RIGHT-HAND-SIDE RHS
      CONSTRNT  LOWER BOUND  CURRENT VALUE  UPPER BOUND
   1.  REF-A      92.000        95.000       104.40
   2.  REF-B      60.000        63.000        73.555
   3.  REF-C     113.00        116.00        UNBOUNDED
   4.  CIT-1      46.660        55.000        58.000
   5.  CIT-2      70.415        73.000        74.229
   6.  CIT-3     103.56        105.00        106.23
   7.  CIT-4      27.445        38.000        41.000
   8.  AP-CIT-1  -.86750        .00000        14.186
   9.  AP-CIT-2  -.20932        .00000        .44014
  10.  AP-CIT-3  -.14516        .00000        .17010
  11.  AP-CIT-4 -1.1658         .00000       UNBOUNDED
  12.  PRICELIM  11.315         11.400        11.568
  13.  ROUTE-A2  15.602         25.000       UNBOUNDED
  14.  ROUTE-B2  14.980         20.000       UNBOUNDED
  15.  ROUTE-B3  14.994         20.000        25.865
  16.  ROUTE-C1 -.23842E-06     25.000       UNBOUNDED

OUTPUT OPTION? NO
```

Figure E3.9   (*Cont.*)

a) The objective-function coefficients changed as follows:

K-T-J from 546 to 550,
L-T-A from 590 to 600.

b) The objective-function coefficients changed as follows:

L-P-J from 580 to 585,
L-T-J from 590 to 585.

c) The objective-function coefficients changed as follows:

L-P-J from 580 to 585,
L-T-J from 590 to 588.

d) The righthand-side values changed as follows:

L-Day-J from 28 to 25,
War-Cap from 20 to 24.

e) The righthand-side values changed as follows:

L-Day-J from 28 to 23,
P-Dem-A from 1400 to 1385.

28. [Excel spreadsheet available at http://web.mit.edu/15.053/www/Exer3.28.xls] Mr. Watson has 100 acres that can be used for growing corn or soybeans. His yield is 95 bushels per acre per year of corn of 60 bushels of soybeans. Any fraction of the 100 acres can be devoted to growing either crop. Labor requirements are 4 hours per acre per year, plus 0.70 hour per bushel of corn and 0.15 hour per bushel of soybeans. Cost of seed, fertilizer, and so on is 24 cents per bushel of corn and 40 cents per bushel of soybeans. Corn can be sold for $1.90 per bushel, and soybeans for $3.50 per bushel. Corn can be purchased for $3.00 per bushel, and soybeans for $5.00 per bushel.

525	580	546	590	−1.0	−1.2	525	580	546	590			
JULY				INV-P	INV-T	AUGUST				Rela-tion	Limit	Constraint name
K-P	L-P	K-T	L-T			K-P	L-P	K-T	L-T			
1	0	1	0	0	0	0	0	0	0	≤	26	K-Days-July
0	1	0	1	0	0	0	0	0	0	≤	28	L-Days-July
40	50	0	0	−1	0	0	0	0	0	≤	1200	P-Dem-July
0	0	35	42	0	−1	0	0	0	0	≤	800	T-Dem-July
0	0	0	0	1	1	0	0	0	0	≤	20	Warehouse
0	0	0	0	0	0	1	0	1	0	≤	26	K-Days-August
0	0	0	0	0	0	0	1	0	1	≤	27	L-Days-August
0	0	0	0	1	0	40	50	0	0	≤	1400	P-Dem-August
0	0	0	0	0	1	0	0	35	42	≤	900	T-Dem-August
1	1	1	1	0	0	1	1	1	1	≤	102	Max-days-total
										=	z (Max)	Contribution

INDEX

K : Kolbert
L : Loomis
P : Plastic
T : Teflon
INV : Inventory
DEM : Demand maximum
Max-days-total : The maximum number of days available for production due to vacation constraints.

Figure E3.10 Formulation of the Krebs Wire linear program.

In the past, Mr. Watson has occasionally raised pigs and calves. He sells the pigs or calves when they reach the age of one year. A pig sells for $80 and a calf for $160. One pig requires 20 bushels of corn or 25 bushels of soybeans, plus 25 hours of labor and 25 square feet of floor space. One calf requires 50 bushels of corn or 20 bushels of soybeans, 80 hours of labor, and 30 square feet of floor space.

Mr. Watson has 10,000 square feet of floor space. He has available per year 2000 hours of his own time and another 4000 hours from his family. He can hire labor at $3.00 per hour. However, for each hour of hired labor, 0.15 hour of his time is required for supervision.

Mr. Watson's son is a graduate student in business, and he has formulated a linear program to show his father how much land should be devoted to corn and soybeans and in addition, how many pigs and/or calves should be raised to maximize profit.

In Fig. 3.12, Tableau 1 shows an initial simplex tableau for Watson's farm using the definitions of variables and constraints given below, and Tableau 2 shows the results of several iterations of the simplex algorithm.

Variables

1. Grow 1 acre of corn	7. Raise 1 pig on corn
2. Grow 1 acre of soybeans	8. Raise 1 pig on soybeans
3. Buy 1 bu. of corn	9. Raise 1 calf on corn
4. Buy 1 bu. of soybeans	10. Raise 1 calf on soybeans
5. Sell 1 bu. of corn	11. Hire 1 hour of labor
6. Sell 1 bu. of soybeans	12–15. Slack variables

Constraints

```
OPTIMAL SOLUTION FOUND.
     CONTRIB      56839.0

OUTPUT OPTION? EXTENDED

ALL ITEMS NOT LISTED IN SECTIONS 1 - 4 HAVE THE VALUE ZERO.

*1* DECISION VARIABLES
   1. K-P-J     26.0000
   2. L-P-J     3.20000
   4. L-T-J     19.5238
   6. INV-T     20.0000
   7. K-P-A     .857143
   8. L-P-A     27.0000
   9. K-T-A     25.1429

*2* SLACK(+) AND SURPLUS(-) IN CONSTRAINTS
   2. +L-DAY-J   5.27619
   8. +P-DEM-A   15.7143
  10. +M-DAY-T   .276191

*3* SHADOW PRICES FOR CONSTRAINTS
   1. K-DAY-J    61.0000
   3. P-DEM-J    11.6000
   4. T-DEM-J    14.0476
   5. WAR-CAP    12.2476
   6. K-DAY-A    525.000
   7. L-DAY-A    580.000
   9. T-DEM-A    .600000

*4* REDUCED COSTS FOR DECISION VARIABLES
   3. K-T-J     -6.66667
   5. INV-P     -1.64762
  10. L-T-A     -15.2000

*5* RANGES ON COEFFICIENTS OF OBJECTIVE CONTRIB
       VARIABLE   LOWER BOUND   CURRENT VALUE   UPPER BOUND
   1. K-P-J        518.33         525.00        UNBOUNDED
   2. L-P-J        .00000         580.00          588.33
   3. K-T-J        UNBOUNDED      546.00          552.67
   4. L-T-J        582.00         590.00        UNBOUNDED
   5. INV-P        UNBOUNDED     -1.0000          .64762
   6. INV-T        -2.8476       -1.2000        UNBOUNDED
   7. K-P-A        467.33         525.00          537.67
   8. L-P-A        564.80         580.00        UNBOUNDED
   9. K-T-A        533.33         546.00          603.67
  10. L-T-A        UNBOUNDED      590.00          605.20

*6* RANGES ON VALUES OF RIGHT-HAND-SIDE CONSTRAI
       CONSTRNT   LOWER BOUND   CURRENT VALUE   UPPER BOUND
   1.  K-DAY-J     19.405         26.000          27.381
   2.  L-DAY-J     22.724         28.000        UNBOUNDED
   3.  P-DEM-J     1040.0         1200.0          1213.8
   4.  T-DEM-J     -20.000        800.00          811.60
   5.  WAR-CAP     .00000         20.000          31.600
   6.  K-DAY-A     25.143         26.000          26.276
   7.  L-DAY-A     .00000         27.000          27.276
   8.  P-DEM-A     1384.3         1400.0        UNBOUNDED
   9.  T-DEM-A     886.25         900.00          930.00
  10.  M-DAY-T     101.72         102.00        UNBOUNDED
```

Figure E3.11   Solution of the Krebs Wire model.

Tableau 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	Relation	RHS
1. Land	1	1									1					=	100
2. Corn	−95		−1		1		20		50							=	0
3. Soybeans		−60		−1		1		25		20						=	0
4. Space							0.25	0.25	0.30	0.30				1		=	100
5. Labor	0.705	0.13					0.25	0.25	0.80	0.80	−0.85	1				=	60
6. Farmer											0.15		1			=	20
$	22.8	24	3.00	5.00	−1.90	−3.50	−80	−80	−160	−160	300					=	0

Tableau 2

	x_1 x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14} x_{15}	Relation	RHS
1. Land	1	−0.0105		0.0105		0.2105		0.5264			1			=	100
2. Corn		−0.00756		0.00756		0.4638	0.3125	1.378	1	−1.065	−0.1625	1.25		=	58.75
3. Soybeans		−0.4803	−1	0.4803	1	3.355	18.75	4.014		21.25	63.25	−25.0		=	4825
4. Space		0.0023		−0.0023		0.1109	0.1562	−0.1134		0.319	0.0488	−0.375	1	=	82.375
5. Labor	1	0.0105		−0.0105		−0.2105		−0.5264						=	0
6. Farmer										0.15		1		=	20
$		0.1212	1.5	0.9788		5.701	35.625	73.94		204.38	171.375	112.50		=	23887.5

Figure E3.12   Initial and final tableaus for Watson's model.

1. Acres of land
2. Bushels of corn
3. Bushels of soybeans

4. Hundreds of sq. ft. floor space
5. Hundreds of labor hours
6. Hundreds of farmer hours

Objective

Dollars of cost to be minimized.

a) What is the optimal solution to the farmer's problem?

b) What are the binding constraints and what are the shadow prices on these constraints? [*Hint*: The initial tableau is not quite in canonical form.]

c) At what selling price for corn does the raising of corn become attractive? At this price +$0.05, what is an optimal basic solution?

d) The farmer's city nephew wants a summer job, but because of his inexperience he requires 0.2 hours of supervision for each hour he works. How much can the farmer pay him and break even? How many hours can the nephew work without changing the optimal basis? What activity leaves the basis if he works more than that amount?

e) One of the farmer's sons wants to leave for the city. How much can the old man afford to pay him to stay? Since that is not enough, he goes, reducing the family labor pool by 2000 hours/year. What is the optimal program now?

f) How much can the selling price of soybeans increase without changing the basis? Decrease? For both of these basis changes, what activity leaves the basis? Are these basis changes intuitively obvious?

g) Does there exist an alternative optimal solution to the linear program? Alternative optimal shadow prices? If so, how can they be found?

29. The initial data tableau, in canonical form, for a linear program to be minimized is given below:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_5	3	1	1	1	1	1		
x_6	2	3	4	1	1		1	
x_7	1	1	3	2	1			1
$(-z)$	0	-4	-6	-1	-3			

A standard commercial linear-programming code employing the *revised simplex method* would have the following information at its disposal: (1) the above initial data tableau; (2) the current values of the basic variables and the row in which each is basic; and (3) the current coefficients of the initial unit columns. Items (2) and (3) are given below:

Basic variables	Current values	x_5	x_6	x_7
x_5	$\frac{12}{5}$	1	$-\frac{2}{5}$	$\frac{1}{5}$
x_1	$\frac{2}{5}$		$\frac{3}{5}$	$-\frac{4}{5}$
x_2	$\frac{1}{5}$		$-\frac{1}{5}$	$\frac{3}{5}$
$(-z)$	$\frac{14}{5}$		$\frac{6}{5}$	$\frac{2}{5}$

a) What is the current basic feasible solution?

b) We can define *simplex multipliers* to be the shadow prices associated with the current basic solution even if the solution is not optimal. What are the values of the simplex multipliers associated with the current solution?

c) The current solution is optimal if the reduced costs of the nonbasic variables are nonnegative. Which variables are nonbasic? Determine the reduced cost of the nonbasic variables and show that the current solution is *not* optimal.

d) Suppose that variable x_4 should now be introduced into the basis. To determine the variable to drop from the basis, we use the minimum-ratio rule, which requires that we know not only the current righthand-side values but also the coefficients of x_4 in the current tableau. These coefficients of x_4 need to be computed.

In performing the simplex method, multiples of the initial tableau have been added to and subtracted from one another to produce the final tableau. The coefficients in the current tableau of the initial unit columns summarize these operations.

i) What multiple of rows 1, 2, and 3, when added together, *must* produce the current row 1 (even though we do not know all the current coefficients in row 1)? The current row 2? The current row 3?

ii) Using the rationale of (i) determine the coefficients of x_4 in the current tableau. Note that it is unnecessary to determine any of the other unknown columns.

iii) How should the pivot operation be performed to update the tableau consisting of only x_5, x_6, and x_7?

You have now completed an iteration of the simplex algorithm using *only* (1) the initial data, (2) the current values of the basic variables and the row in which each is basic, and (3) the current coefficients of the initial unit columns. This is the essence of the revised simplex method. (See Appendix B for further details.)

Exercises 13 and 28 are variations of problems used by C. Roger Glassey of the University of California, Berkeley, and Exercise 28 is in turn based on a formulation exercise from *Linear Programming*, by G. Hadley, Addison-Wesley Pyblishing Company, Inc., 1963.

Exercise 17 is inspired by the French Electric Power Industry case written by John E. Bishop, based on ''Application of Linear Programming to Investments in the Electric Power Industry,'' by P. Mass*é* and R. Gibrat, which appeared in *Management Science*, 1957.

Exercise 25 is based on the Holden Consulting Company case written by Basil A. Kalyman.

Exercise 26 is based on the Krebs Wire Company case written by Ronald S. Frank, based on a case of one of the authors.